UNIVERSITY OF SURREY©

B. Sc. Undergraduate Programmes in Mathematical Studies
M. Math. Undergraduate Programmes in Mathematical Studies

Level HE2 Examination

Module MS211 NUMERICAL AND COMPUTATIONAL METHODS

Time allowed – $1\frac{1}{2}$ hrs Spring Semester 2006

Attempt THREE questions.
If any candidate attempts more than THREE questions, only the best THREE solutions will be taken into account.

Question 1

(a) (i) Consider the linear system $A\mathbf{x} = \mathbf{b}$ where A is nonsingular. Suppose that $\tilde{\mathbf{x}}$ is an approximate solution to this system which satisfies $A\tilde{\mathbf{x}} = \tilde{\mathbf{b}}$. Show that

$$\frac{||\mathbf{x} - \tilde{\mathbf{x}}||}{||\mathbf{x}||} \leq ||A||\ ||A^{-1}||\ \frac{||\mathbf{b} - \tilde{\mathbf{b}}||}{||\mathbf{b}||}.$$

(ii) Show that the condition number $K(A) = ||A||\ ||A^{-1}||$ satisfies $K(A) \geq 1$ for all matrices A assuming that a natural matrix norm is used.

(iii) What is the significance of the condition number? [10]

(b) Consider the linear system

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$$

The Gauss-Seidel iteration for solving this system can be expressed in the form

$$\mathbf{x}^{(k+1)} = M\mathbf{x}^{(k)} + \mathbf{c}.$$

Find M and $\mathbf{c}$.

State a condition for this iteration to converge for any $\mathbf{x}_0$ and determine whether or not the Gauss-Seidel iteration converges for the linear system above. [10]

Question 2

(a) Consider the following iteration for some constant $\beta > 0$:

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{\beta}{x_n}\right)$$

(i) Show that $\sqrt{\beta}$ is a fixed point of the iteration. [2]

(ii) Show that the iteration converges quadratically to this fixed point. [6]

(iii) Taking $\beta = 2$ and $x_0 = 1.5$, compute the first 2 iterates. Use these values to show that the error is decreasing quadratically and calculate the asymptotic error constant. [6]

(b) State Newton's method for solving the system of nonlinear equations $\mathbf{f}(\mathbf{x}) = \mathbf{0}$.

Rewrite the Newton iteration as a two-stage iteration and explain why this form of the iteration should be used in practice. [6]

Question 3

(a) Let $p_n(x)$ be the polynomial of degree n which interpolates the function $f(x)$ at the points x_i, $i = 0, 1, \ldots, n$ in the interval $[a, b]$.

For each $x \in [a, b]$

$$f(x) = p_n(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!}\prod_{i=0}^{n}(x - x_i) \qquad (1)$$

for some $\xi \in (a, b)$. Prove this result for $n = 1$. [10]

(b) Use equation (1) to show that on the interval $[x_0, x_1]$,

$$|f(x) - p_1(x)| \leq \frac{1}{8}(x_1 - x_0)^2 M,$$

where $M = \max_{\xi \in [x_0, x_1]} |f''(\xi)|$. [4]

(c) Use the divided difference form of the interpolating polynomial to find the quadratic polynomial which interpolates the following data:

$$x_0 = 1, \quad x_1 = 3, \quad x_2 = 4,$$

$$f_0 = 3, \quad f_1 = 7, \quad f_2 = 12.$$

Hence find an approximation to $f(2)$. [6]

Question 4

(a) State the three different finite difference approximations to the derivative $f'(x_0)$, together with the order of their truncation errors.

Derive the forward difference approximation and its leading error term [9]

(b) Gaussian quadrature approximates the integral

$$\int_a^b f(x)dx$$

by the sum

$$\sum_{k=0}^{n} c_k f(x_k).$$

The unknown coefficients and points are found by requiring that

$$\int_a^b x^j dx = \sum_{k=0}^{n} c_k x_k^j, \quad j = 0, 1, \ldots, 2n+1.$$

(i) Use this to derive four equations to be solved for the two point quadrature rule

$$\int_{-1}^{1} f(x)dx \approx a_0 f(x_0) + a_1 f(x_1)$$

which is exact when f is a polynomial of degree 3 or less. Show that

$$a_0 = a_1 = 1, \quad x_0 = -\frac{1}{\sqrt{3}}, \text{ and } x_1 = \frac{1}{\sqrt{3}}.$$

is the solution of these equations. [7]

(ii) Find the 2 point Gaussian quadrature approximation to the integral

$$\int_0^2 \sin\left(\frac{3\pi}{2}(x-1)^2\right) dx.$$

[4]

Question 5

(a) A multistep method for solving the differential equation

$$\dot{y} = f(t, y), \qquad y(0) = \alpha$$

can be derived by integrating the equation over the interval $[t_{i-1}, t_{i+1}]$ and then approximating $f(t, y(t))$ by $P_1(t)$, where $P_1(t)$ is a linear polynomial which interpolates $f(t, y(t))$ at the points t_i and t_{i-1}. We define $t_i = ih$ where $i = 0, 1, 2, \ldots$ and h is the step size.

Show that the method derived in this way is given by

$$y_{i+1} = y_{i-1} + 2hf(t_i, y_i).$$

You may use the following results without verifying them:

$$\int_{t_{i-1}}^{t_{i+1}} (t - t_i)\, dt = 0, \qquad \int_{t_{i-1}}^{t_{i+1}} (t - t_{i-1})\, dt = 2h^2.$$

Hint: Use $t_i - t_{i-1} = h$. [8]

(b) Describe the linear shooting method for solving the boundary value problem

$$y'' - xy' - x^2 y = 2x^3, \qquad 0 \le x \le 1,$$

$$y(0) = 1, \qquad y(1) = -1,$$

which involves the solution of two initial value problems for y_1 and y_2.

Rewrite the two initial value problems which have to be solved as a first order system of equations.

State Euler's method applied to this first order system. [12]

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