# UNIVERSITY OF SURREY©

**B. Sc. Undergraduate Programmes in Mathematical Studies**
**M. Math. Undergraduate Programmes in Mathematical Studies**

**Level HE1 Examination**

Module MS132 PROBABILITY AND STATISTICS

Time allowed – 2 hours Spring Semester 2006

Attempt FOUR questions. If any candidate attempts more than FOUR questions only the best FOUR solutions will be taken into account.
Cambridge Statistical Tables and a formula sheet will be provided.

**Question 1**

(a) For married couples living in a certain region the probability that the husband will vote on a referendum is 0.24, the probability that the wife will vote is 0.31 and the probability that they will both vote is 0.16. What is the probability that;

(i) At least one member of a married couple will vote?

(ii) a husband will vote, given that his wife will vote?

(iii) a wife will not vote, given that her husband will vote? [9]

State with justification whether or not the probability that a husband will vote is independent of the probability that his wife will vote. [2]

(b) (i) A committee of size 8 is selected at random from 12 professional people and 28 non-professional people.

Write down an expression for the probability that exactly 3 members of the committee are professional people and evaluate this probability.
[ Give your answer correct to four decimal places] [4]

(ii) A committee of size 8 is selected at random from 1200 professional people and 2800 non-professional people.

i. Write down the expression for the probability that exactly 3 members of the committee are professional people and evaluate this probability.
[ Give your answer correct to four decimal places] [4]

ii. Write down a binomial random variable that can be used to approximate the probability that exactly 3 members of the committee are professional people and evaluate this probability.
[ Give your answer correct to four decimal places] [6]

**Question 2**

(a) A random variable $Y$ has an exponential distribution with parameter $\lambda$.

Show that $\mathrm{E}[Y] = \frac{1}{\lambda}$ and $\mathrm{Var}[Y] = \frac{1}{\lambda^2}$. [5]

(b) A random variable $X$ has an exponential distribution with parameter 4.

(i) Using part (a) write down the mean of $X$. [1]

(ii) Show that the median of $X$ is $\frac{1}{4}\ln 2$. [4]

(iii) Calculate the probability that an observation selected at random from $X$ will lie between the median and the mean. [3]

(iv) Calculate the probability that if five observations are selected at random then exactly two will lie between the median and the mean. [4]

(c) Three random variables $U$, $V$ and $W$ are independent exponential random variables with parameters $\lambda_U$, $\lambda_V$ and $\lambda_W$.

Find the following in terms of $\lambda_U$, $\lambda_V$ and $\lambda_W$:

(i) $\mathrm{E}[U + V + W]$; [1]

(ii) $\mathrm{E}[UV]$; [2]

(iii) $\mathrm{Var}[U + V + W]$; [2]

(iv) $\mathrm{Var}[2V - 3W]$. [3]

## Question 3

(a) The probability mass function of a Poisson distribution with mean $\mu$ is given by;

$$P(X = x) = \frac{\mu^x e^{-\mu}}{x!}, \qquad x = 0, 1, 2, \ldots.$$

Let $x_1, x_2, \ldots, x_n$ be $n$ independent observations from this distribution.

Derive the Maximum Likelihood estimator and the Method of Moments estimator for $\mu$. [9]

(b) The number of faults reported during the warranty period on each of 90 cars sold by a particular vehicle dealer were recorded.

The data are shown in the following table.

| Number of faults | 0 | 1 | 2 | 3 | 4 | 5 | 6+ |
|---|---|---|---|---|---|---|---|
| Frequency | 23 | 29 | 20 | 10 | 4 | 4 | 0 |

(i) Test whether it is reasonable to assume that the data follow a Poisson distribution with mean 1. [7]

(ii) Assuming that the data in the table do follow a Poisson distribution, obtain the Maximum Likelihood estimator for the mean $\mu$. [2]

(iii) Test whether the data follow a Poisson distribution with unspecified mean. [7]

**Question 4**

(a) Outline the situations in which the following tests would be used. In each case state clearly the assumptions made in using the test.

(i) A two-sample $t$-test.

(ii) A paired $t$-test. [6]

(b) (i) A variety of coffee is packaged using two machines $X$ and $Y$. A random samples of 9 packets of coffee was taken from $X$ and a random samples of 16 packets of coffee was taken from $Y$.

The masses in grams of the packets from machine $X$ were recorded as $x_i$, and those from machine $Y$ were recorded as $y_j$, with $i = 1, 2, \ldots, 9$, and $j = 1, 2, \ldots, 16$.

Summary statistics are given below:

$$\bar{x} = 30.1, \qquad \textstyle\sum_{i=1}^{9}(x_i - \bar{x})^2 = 0.801,$$

$$\bar{y} = 29.6, \qquad \textstyle\sum_{j=1}^{16}(y_j - \bar{y})^2 = 3.021.$$

Test whether there is a significant difference between the masses of packets of coffee from the two machines. [9]

(ii) Eight athletes ran a 400 metres race at sea level and at a later meeting ran another 400 metres race at high altitude.

Their times in seconds for the two races were as follows:

| Runner | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| Time at sea level(s) | 48.3 | 47.6 | 49.2 | 50.3 | 48.8 | 51.1 | 49.0 | 48.1 |
| Time at high altitude(s) | 50.4 | 47.3 | 50.8 | 52.3 | 47.7 | 54.5 | 48.9 | 49.9 |

Test the hypothesis that the athletes' performance is not affected by the altitude.

[10]

**Question 5**

A biased coin has probability $\frac{1}{4}$ of landing heads up when tossed.

The coin is tossed three times. The result of the first toss is scored as $X_1 = 1$ for heads and $X_1 = 0$ for tails; $X_2$ and $X_3$ are scored likewise for the second and third tosses.

Let $Y_1 = X_1 + X_2 + X_3$ and $Y_2 = X_1 - X_2$.

Show that $\mathrm{P}(Y_1 = 2, Y_2 = 0) = \frac{3}{64}$. Calculate the remaining probabilities in the bivariate distribution of the pair $(Y_1, Y_2)$ and display the joint probabilities in an appropriate table. [6]

(a) Find the marginal probability distributions of $Y_1$ and $Y_2$. [4]

(b) Calculate the means and variances of $Y_1$ and $Y_2$. [5]

(c) Calculate the covariance of $Y_1$ and $Y_2$. Comment on your result. [4]

(d) Find the conditional distribution of $Y_1$ given $Y_2 = 1$. [4]

(e) Find the conditional mean of $Y_1$ given $Y_2 = 1$. [2]

**Question 6**

(a) For a random variable $X$, the moment generating function $M_X(z)$ is defined as $E[e^{zX}]$. Show that

$$\frac{dM_X(0)}{dz} = E[X] \quad \text{and} \quad \frac{d^2M_X(0)}{dz^2} = E[X^2].$$

Hence find an expression for $\text{Var}[X]$. [5]

(b) A random variable $X$ has probability density function

$$f(x) = \begin{cases} kxe^{-3x}, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

(i) Find the value of $k$. [4]

(ii) Show that the moment generating function of $X$ is given by:

$$M_X(z) = \frac{9}{(3-z)^2}.$$

[6]

(iii) Using the moment generating function, calculate $E[X]$, $\text{Var}[X]$ and show that $E[X(X+1)(X+2)] = \frac{38}{9}$. [10]

**INTERNAL EXAMINER: J.D. Godolphin**