UNIVERSITY OF SURREY©

B. Sc. Undergraduate Programmes in Mathematical Studies
M. Math. Undergraduate Programmes in Mathematical Studies

Level HE1 Examination

Module MS124 LINEAR ALGEBRA

Time allowed – 2 hours

Spring Semester 2008

Answer any **four** of the six questions.

If you attempt more than four questions, only your BEST FOUR answers will be taken into account.

Each question carries 25 marks.

Approved calculators may be used.

Question 1

(a) For each of the following, give the definition and one 3×3 example *other than* the zero or identity matrix.

(i) A **symmetric** matrix. [2]

(ii) A **singular** matrix. [2]

(iii) An **orthogonal** matrix. [3]

(b) Let A be the matrix $\begin{pmatrix} 1 & 3 & 2 & 0 & 5 \\ 2 & 0 & 1 & -1 & 3 \\ 3 & -3 & 0 & -2 & 1 \\ 1 & 9 & 5 & 1 & 12 \end{pmatrix}$.

(i) Find a basis for the row-space of A, [3]

(ii) State the rank of A. [1]

(iii) Without further calculations, write down a basis for the column-space of A. [3]

(iv) Find a basis for the null-space of A. [7]

(v) State the nullity of A. [1]

(vi) Express $\mathbb{R}^5$ as the direct sum of two orthogonal subspaces. [3]

Question 2

(a) Give the full definition of a **real vector space** [8]

(b) Let $v_1, \ldots, v_n$ be elements of a real vector space V.

(i) State what is meant by the **span** of $\{v_1, \ldots, v_n\}$. [2]

(ii) Show that $\text{span}\{v_1, \ldots, v_n\}$ is a subspace of V. [5]

(c) Let X be a fixed $n \times n$ matrix and let U be the set of all $n \times n$ matrices A with the property that $\mathrm{AX} = \mathrm{XA}^t$.

(i) Show that U is a subspace of $M_{n,n}(\mathbb{R})$. [5]

(ii) Find a basis for U when $n = 2$ and $\mathrm{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. [5]

Question 3

(a) Let V and W be real vector spaces.

State what is meant by a **linear map** from V to W. [4]

(b) Let U be the subspace of $\mathcal{F}(\mathbb{R})$ which consists of all twice-differentiable functions of a real variable t.

Let $S : U \to \mathcal{F}(\mathbb{R})$ be defined by $S\ :\ \mathrm{f}(t) \mapsto \mathrm{f}''(t) + 4\,\mathrm{f}(t)$.

(i) Show that S is a linear map. [4]

(ii) By solving a second-order differential equation, find the kernel of S. [4]

(c) Let V be a real vector space with ordered basis $\alpha = (v_1, v_2, v_3)$.

Let T be the linear map of V defined by its effect on the basis elements as follows:

$$Tv_1 = v_1 - v_2,\ Tv_2 = v_1 + v_2 + v_3,\ Tv_3 = 2v_1 + 4v_2 + 3v_3.$$

(i) Write down the matrix A which represents T relative to α. [2]

(ii) If β is the ordered basis $(v_1,\ v_1 + v_2,\ v_1 + v_2 + v_3)$ for V, write down the change matrix from β-coordinates to α-coordinates. [3]

(iii) Hence find the matrix B which represents T relative to β. [6]

(iv) Give the mathematical name for the relationship between A and B. [2]

Question 4

(a) Let T be a linear map of a vector space V.

State what is meant by an **eigenvalue** of T. [3]

(b) Let λ and μ be distinct eigenvalues of a linear map T and let V_λ, V_μ be the corresponding eigenspaces. Prove that $V_\lambda \cap V_\mu = \{0\}$. [5]

(c) T is the linear map of $\mathbb{R}^3$ whose standard matrix is $\mathrm{A} = \begin{pmatrix} -2 & -2 & 1 \\ -2 & 1 & -2 \\ 1 & -2 & -2 \end{pmatrix}$.

You are given that the eigenvalues of T are -3 and 3. The eigenspace V_3 is one-dimensional, with basis $\{(1, -2, 1)\}$.

(i) Find a basis for the eigenspace V_{-3}. [6]

(ii) Find a diagonal matrix D and an orthonormal ordered basis α for V such that D is the matrix of T relative to α. [6]

(iii) Transform the quadratic form $-2{x_1}^2 + {x_2}^2 - 2{x_3}^2 - 4x_1x_2 + 2x_1x_3 - 4x_2x_3$ into the form $a_1{y_1}^2 + a_2{y_2}^2 + a_3{y_3}^2$, giving the values of a_1, a_2, a_3 and expressing each of y_1, y_2 and y_3 in terms of x_1, x_2 and x_3. [5]

Question 5

(a) State what is meant by saying that an $n \times n$ real matrix is **negative definite**. [3]

(b) Let N be a negative definite $n \times n$ real matrix and let I be the $n \times n$ identity matrix. Prove that the matrix N − I is negative definite. [5]

(c) Let A be the matrix $\begin{pmatrix} -3 & 2 \\ 2 & -3 \end{pmatrix}$.

(i) Show that A is negative definite. [4]

(ii) Find the matrix exp(A). [9]

(iii) Hence find the solution of the system of ordinary differential equations

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = -3x_1 + 2x_2, \quad \frac{\mathrm{d}x_2}{\mathrm{d}t} = 2x_1 - 3x_2$$

for which $x_1 = x_2 = 1$ when $t = 0$. [4]

Question 6

(a) Let U be the subspace of $\mathbb{R}^5$ with basis $\{(1,0,1,0,0),(0,1,2,1,0),(5,2,1,2,1)\}$.

Use the Gram-Schmidt process to find an orthonormal basis for U. [8]

(b) Let A and B be $n \times n$ real matrices and define $\langle \mathrm{A}, \mathrm{B} \rangle$ to be the trace of $\mathrm{A}^t\mathrm{B}$.

Given that this defines a positive definite bilinear form on $M_{n,n}(\mathbb{R})$, show that it is an inner product and state the value of $\langle \mathrm{A}, \mathrm{A} \rangle$ when A is an orthogonal matrix. [5]

(c) (i) State what is meant by a **Hermitian matrix**. [2]

(ii) Let H be a 2×2 Hermitian matrix. Show that det(H) is a real number. [5]

(d) Using the standard inner product on $\mathbb{C}^2$, evaluate

(i) $\langle\, (3\mathrm{i},\, 2-\mathrm{i}), (2+\mathrm{i}, -\mathrm{i})\,\rangle$, [3]

(ii) the norm of (i, i). [2]

INTERNAL EXAMINER: Dr D.J. Fisher