UNIVERSITY OF SURREY©

B. Sc. Undergraduate Programmes in Mathematical Studies
M. Math. Undergraduate Programmes in Mathematical Studies

Level HE1 Examination

Module MS113 LINEAR ALGEBRA

Time allowed – 2 hrs

Spring Semester 2007

Answer any **four** of the six questions.

If you attempt more than four questions, only your BEST FOUR answers will be taken into account.

Each question carries 25 marks.

Question 1

(a) Let U be a subspace of $\mathbb{R}^n$. Give definitions of

(i) a **basis** for U, [3]

(ii) the **dimension** of U, [2]

(iii) a **complement** of U in $\mathbb{R}^n$. [3]

(b) S is the subset $\{(1,-2,1,4),(3,-1,2,3),(2,6,0,-10)\}$ of $\mathbb{R}^4$.

(i) Show that S is a linearly dependent set and find a basis for the subspace of $\mathbb{R}^4$ spanned by S. [7]

(ii) Hence find a basis for a complement of span(S) in $\mathbb{R}^4$. [2]

(c) Let $\{\mathbf{v}_1,\mathbf{v}_2,\dots,\mathbf{v}_m\}$ be a linearly independent set of vectors in $\mathbb{R}^n$.

(i) Briefly explain why $m \le n$. You may assume any result that was proved in the lectures. [2]

(ii) Show that if the set $\{\mathbf{v}_1,\mathbf{v}_2,\dots,\mathbf{v}_m,\mathbf{u}\}$ is linearly dependent then $\mathbf{u}$ is a linear combination of the vectors $\mathbf{v}_1,\mathbf{v}_2,\dots,\mathbf{v}_m$. [6]

Question 2

Let U and V be subspaces of $\mathbb{R}^n$.

(a) Prove that the intersection $U \cap V$ is a subspace of $\mathbb{R}^n$. [8]

(b) Now let $U = \text{span}\{(1,0,1),(0,1,0)\}$ and $V = \text{span}\{(1,1,0)\}$.
Show that $U \cap V = \{(0,0,0)\}$ and deduce that $\mathbb{R}^3 = U \oplus V$. [7]

Let $\alpha = (\mathbf{i},\mathbf{j},\mathbf{k})$ be the standard ordered basis for $\mathbb{R}^3$ and let β be the ordered basis $(\,(1,0,1),(0,1,0),(1,1,0)\,)$.

(c) Write down the change matrix P from β to α coordinates and find the change matrix from α to β coordinates. [7]

(d) Let A and B be 3×3 matrices such that PB = AP, where P is the change matrix found in part (c). Describe the significance of this relationship in terms of a linear map of $\mathbb{R}^3$. [3]

Question 3

(a) For each of the following, state whether or not it is one of the axioms for a real vector space V. [4]

(i) $(u + v) + w = u + (v + w)$ for all $u, v, w \in V$,

(ii) there exists an element $1 \in V$ such that $1v = v$ for all $v \in V$,

(iii) $uv = vu$ for all $u, v \in V$,

(iv) $(a + b)v = av + bv$ for all $a, b \in \mathbb{R}$ and $v \in V$.

(b) Let X be a fixed 2×2 matrix. Let Z_{X} be the set of 2×2 real matrices A such that XA is the 2×2 zero matrix.

(i) Prove that Z_{X} is a subspace of $M_{2,2}(\mathbb{R})$. [6]

(ii) Find a basis for Z_{X} when $\mathrm{X} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. State the dimension of Z_{X} in this case. [8]

(c) V and W are vector spaces with ordered bases $\alpha = (v_1, v_2, v_3)$ and $\beta = (w_1, w_2, w_3, w_4)$ respectively.

$T : V \to W$ is the linear map defined by its effect on the basis elements as follows:

$$Tv_1 = w_1 + w_2 + w_3 + w_4,\ Tv_2 = w_1 - w_2 - w_3,\ Tv_3 = w_1 + 3w_2 + 3w_3 + 2w_4.$$

Find a basis for the kernel of T. [7]

Question 4

(a) If λ is an eigenvalue of a linear map $T : V \to V$, show that $\lambda + 1$ is an eigenvalue of $T + I$ where I denotes the identity linear map on V. [5]

(b) T is the linear map of $\mathbb{R}^3$ whose standard matrix is $\mathrm{A} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

(i) Show that T has two distinct eigenvalues. [7]

(ii) Find bases for the two eigenspaces of T. [10]

(iii) Write down an ordered basis of $\mathbb{R}^3$ consisting of eigenvectors of T. Give the matrix which represents T relative to this ordered basis. [3]

Question 5

(a) Let q be the real quadratic form $3x^2 - 4xy + 6y^2$.

(i) Find an orthogonal change of coordinates which transforms q to the form $aX^2 + bY^2$. State the values of a and b, and give each of X and Y in terms of x and y. [9]

(ii) Hence find the principal axes of the conic $3x^2 - 4xy + 6y^2 = 5$. Sketch the curve and state what type of conic it is. [5]

(b) Let A be a positive definite real matrix. Prove that the matrix $\mathrm{A}^t\mathrm{A}$ is symmetric and positive definite. [6]

(c) Let V be a two-dimensional real vector space with basis $\{u, v\}$.

Let V^* be the vector space consisting of all linear maps from V to $\mathbb{R}$.

(i) Give the word which is usually used to describe the space V^*. [2]

(ii) Write down an ordered basis α for V^*, defining each element clearly. [3]

Question 6

(a) Let U be the subspace of $\mathcal{F}(\mathbb{R})$ spanned by $\{\mathrm{e}^x, \mathrm{e}^{-x}\}$.

Let $B : U \times U \to \mathbb{R}$ be given by $B(\mathrm{f}(x), \mathrm{g}(x)) = \displaystyle\int_0^1 \mathrm{f}(x)\mathrm{g}(x)\ \mathrm{d}x$ for any $\mathrm{f}(x),\ \mathrm{g}(x) \in U$.

(i) Show that B is an inner product on U. [9]

(ii) Find the matrix of B relative to the ordered basis $(\mathrm{e}^x, \mathrm{e}^{-x})$. [7]

(b) (i) State what is meant by a **unitary matrix**. [2]

(ii) Suppose $\begin{pmatrix} r\mathrm{e}^{\mathrm{i}\theta} & r\mathrm{e}^{\mathrm{i}\theta} \\ r\mathrm{ie}^{-\mathrm{i}\theta} & -r\mathrm{ie}^{-\mathrm{i}\theta} \end{pmatrix}$ is a unitary matrix for all real values of θ.

Find the possible values of the real number r. [7]

[You may assume that $\overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}$ for all $z_1, z_2 \in \mathbb{C}$, $\bar{\mathrm{i}} = -\mathrm{i}$ and $\overline{\mathrm{e}^{\mathrm{i}\theta}} = \mathrm{e}^{-\mathrm{i}\theta}$.]

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