UNIVERSITY OF SURREY©

B. Sc. Undergraduate Programmes in Mathematical Studies

Level HE1 Examination

MS113 LINEAR ALGEBRA

Time allowed - 2 hours   Spring Semester 2006

Answer any **four** of the six questions.

If you attempt more than four questions, only your BEST FOUR answers will be taken into account.

Each question carries 25 marks.

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Question 1

(a) Let $v_1, \ldots, v_m$ be elements of a real vector space V. State what is meant by a **linear combination** of $v_1, \ldots, v_m$. [2]

Show that $(1, 2, 4)$ is *not* a linear combination of $(1, 1, 1)$ and $(1, -1, 3)$. [5]

(b) Let A be an $m \times n$ real matrix. Define the **null-space** of A and prove that it is a subspace of $\mathbb{R}^n$. [8]

(c) Find, in parametric form, the solution set of the equations

$$x_1 + 2x_2 + 4x_3 + 3x_4 = 3, \;\; 2x_1 - x_2 - 2x_3 + x_4 = 1.$$ [7]

(d) Using your answer to part (c), write down the null-space of the matrix $\begin{pmatrix} 1 & 2 & 4 & 3 \\ 2 & -1 & -2 & 1 \end{pmatrix}$, giving your answer as the span of a set of vectors. [3]

Question 2

(a) Write down any *four* of the axioms for a **real vector space** V. [4]

(b) State what is meant by the **dimension** of a vector space. [2]

Give an example of a vector space which is not finite-dimensional. [2]

(c) α is the ordered basis $(1, x, x^2, x^3)$ for $P_3(\mathbb{R})$.

Write down the α-coordinates of

(i) $1 + x - x^3$, (ii) $2 + 2x + x^2$, (iii) $1 + x + 2x^2 + 3x^3$. [3]

Find the dimension of the subspace of $P_3(\mathbb{R})$ spanned by $1 + x - x^3$, $2 + 2x + x^2$ and $1 + x + 2x^2 + 3x^3$. [5]

(d) Let V be an n-dimensional vector space and let $\{u_1, \ldots, u_n\}$ be any linearly independent subset of V.

(i) Prove that every element of V is a linear combination of $u_1, \ldots, u_n$. (You may assume that any set which contains *more* than n elements of V is linearly dependent.) [7]

(ii) Deduce that $\{u_1, \ldots, u_n\}$ is a basis for V. [2]

Question 3

(a) Give examples of the following:

(i) Two singular 2×2 matrices whose sum is non-singular. [2]

(ii) A 3×3 matrix with rank 1, whose entries are all non-zero. [2]

(iii) A spanning set for $\mathbb{R}^3$ which is not a basis for $\mathbb{R}^3$. [2]

(iv) Two subspaces of $\mathbb{R}^3$ whose union is not a subspace of $\mathbb{R}^3$. [2]

(v) A real vector space, other than $\mathbb{R}^4$, which is isomorphic to $\mathbb{R}^4$. [2]

(b) U and W are subspaces of a vector space V.

(i) Define the **sum** $U + W$ and prove that it is a subspace of V. [8]

(ii) If $U = \text{span}\{(1,1,1,1,1), (0,1,2,1,2), (0,0,1,1,3)\}$ and $W = \text{span}\{(1,2,3,4,5), (0,1,2,3,4)\}$ are subspaces of $\mathbb{R}^5$, find a basis for $U + W$. [5]

Hence state, with a reason, whether or not $\mathbb{R}^5$ is the direct sum of U and W. [2]

Question 4

(a) S is the linear map of $\mathbb{R}^2$ whose standard matrix is $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.

(i) Find the image under S of the unit square with vertices at $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$. Hence describe in words the geometrical effect of S. [7]

(ii) State the area scale factor of the linear map S. [2]

(iii) If $\mathbf{u} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$, find the images of $\mathbf{u}$ and $\mathbf{v}$ under S. Hence find the matrix which represents S relative to the ordered basis $(\mathbf{u}, \mathbf{v})$ for $\mathbb{R}^2$. [6]

(b) The linear map $T : \mathbb{R}^4 \to \mathbb{R}^3$ is represented, relative to the standard ordered bases, by the matrix $\begin{pmatrix} 2 & 0 & 4 & 2 \\ 0 & 1 & 3 & -5 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

(i) Find bases for the kernel and the image of T. [7]

(ii) State, with reasons, whether T is (i) injective, (ii) surjective. [3]

Question 5

(a) Let V be a real vector space. State what is meant by an **eigenvalue** and a corresponding **eigenvector** of a linear map $T : V \to V$. [3]

(b) Let A be the matrix $= \begin{pmatrix} 4 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 2 & 0 \end{pmatrix}$.

(i) Find and solve the characteristic equation of A. [5]

(ii) Given that $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$ are three linearly independent eigenvectors of A, find an orthogonal matrix P and a diagonal matrix D such that $\mathrm{P}^t\mathrm{AP} = \mathrm{D}$. [7]

(iii) Hence transform the quadratic form $4{x_1}^2 + 3{x_2}^2 + 4x_2x_3$ into the form $a{y_1}^2 + b{y_2}^2 + c{y_3}^2$ where a, b, c are real constants to be found. Express each of y_1, y_2 and y_3 in terms of x_1, x_2 and x_3. [5]

(c) Suppose S and T are linear maps of a vector space V, and $v \in V$ is an eigenvector of both S and T.

Prove that v is an eigenvector of the composite linear map ST. [5]

Question 6

(a) If $\mathrm{B} = \begin{pmatrix} 2 & 6 & 3 \\ 3 & 2 & -6 \\ 6 & -3 & 2 \end{pmatrix}$, find $\mathrm{B}^t\mathrm{B}$. Hence find the value of k for which the matrix $k\mathrm{B}$ represents an isometry of $\mathbb{R}^3$. [5]

(b) U is the subspace of $\mathbb{R}^4$ with basis $\{(3, 1, -1, 3), (5, -1, -5, 7), (1, 1, -2, 8)\}$

Use the Gram-Schmidt process to find a basis for U which is orthonormal relative to the standard inner product on $\mathbb{R}^4$. [8]

(c) Prove that if $\langle u, v \rangle$ is an inner product on a Euclidean space V then $|| u + v || \leq || u || + || v ||$ for all $u, v \in V$. [8]

[Hint: consider $|| u + v ||^2$. You may assume the Cauchy-Schwartz inequality $|\langle u, v \rangle| \leq || u ||.|| v ||$.]

(d) Use the result from part (c) to show that $|| w - v || \geq || w || - || v ||$ for all $v, w \in V$. [4]

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