UNIVERSITY OF SURREY©

B. Sc. Undergraduate Programmes in Mathematical Studies
M. Math. Undergraduate Programmes in Mathematical Studies

Level HE1 Examination

Module MS102 Vector Calculus

Time allowed – 2 hrs

Spring Semester 2007

Attempt FOUR questions. If any candidate attempts more than FOUR questions only the best FOUR solutions will be taken into account.

Question 1

(a) Let $\mathbf{a} = (-1, 4, 3)$ and $\mathbf{b} = (8, 4, -3)$. Find $\mathbf{a} \cdot \mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$. [7]

(b) Find the area of the triangle with vertices at the points $(2, 4, 5)$, $(1, 5, 7)$ and $(-1, 6, 8)$. [9]

(c) Let $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$. Show that

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{i}) = a_2 b_3 - a_3 b_2$$

and find similar formulae for $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{j})$ and $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{k})$. Deduce that

$$\mathbf{a} \times \mathbf{b} = (\mathbf{a} \cdot (\mathbf{b} \times \mathbf{i}))\, \mathbf{i} + (\mathbf{a} \cdot (\mathbf{b} \times \mathbf{j}))\, \mathbf{j} + (\mathbf{a} \cdot (\mathbf{b} \times \mathbf{k}))\, \mathbf{k}.$$ [9]

Question 2

(a) In three dimensions, two straight lines have equations

$$\mathbf{r} = (-1, -1, -1) + t\,(2, 3, 4) \qquad \text{and} \qquad \mathbf{r} = (2, 4, -1) + s\,(1, 2, -4).$$

Prove that these lines intersect and find their point of intersection. [7]

Find, in cartesian form, the equation of the plane which contains both of these lines. [9]

(b) Find a vector normal to the surface

$$z = x^2 - xy - y^2$$

at the point $(1, 1, -1)$. Also find the equation of the tangent plane to the surface at this point, expressing the answer in cartesian form. [9]

Question 3

(a) Let

$$f(x,y) = \frac{1}{x} + \frac{4}{y} + \frac{9}{4-x-y}$$

Find the first and second order partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}$ and $\frac{\partial^2 f}{\partial x \partial y}$.

If we restrict to $x, y > 0$ show that $(x,y) = (\frac{2}{3}, \frac{4}{3})$ is a stationary point of $f(x,y)$ and classify it. [14]

(b) Use Lagrange multipliers to determine the length and breadth of the rectangle of largest area that can be fitted inside the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

with all four corners touching the ellipse and the sides of the rectangle being parallel to the coordinate axes. [11]

Question 4

(a) By converting to polar coordinates, show that

$$\iint_D \sqrt{x^2+y^2+1}\, dx\, dy = \frac{\pi}{6}(2\sqrt{2}-1)$$

where D is the portion of the disk $x^2 + y^2 \le 1$ that lies in the first quadrant. [NB: you must include the calculation of the Jacobian]. [13]

(b) Evaluate $\iint_D \left(\frac{x-y}{x+y}\right) dA$, where D is the region enclosed by the straight lines $y = x$, $y = x - 1$, $y = -x + 1$ and $y = -x + 3$. [Hint: transform to the variables u and v defined by $u = x + y$ and $v = x - y$]. [12]

Question 5

Spherical polar coordinates (r, θ, ϕ) are defined by

$$\begin{aligned} x &= r\sin\theta\cos\phi, \\ y &= r\sin\theta\sin\phi, \\ z &= r\cos\theta. \end{aligned}$$

(a) Draw a diagram to illustrate the geometrical meaning of r, θ and ϕ. [5]

(b) Show that the Jacobian of the transformation from (x, y, z) into (r, θ, ϕ) is $r^2 \sin\theta$. [10]

(c) Suppose $b > a$. Let V be the region consisting of all points that are between the spheres $r = a$ and $r = b$ and above the (x, y) plane. Show that

$$\iiint_V z \exp((x^2+y^2+z^2)^2)\, dV = \frac{\pi}{4}(\exp(b^4) - \exp(a^4)).$$ [10]

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Question 6

(a) State Green's theorem and deduce from it the result that if a region in the (x, y) plane is bounded by a closed curve C then the area of the region is

$$\tfrac{1}{2}\int_C (-y\,dx + x\,dy) \qquad [9]$$

where the curve is traversed anticlockwise.

(b) Suppose that C is the line segment from the point (x_1, y_1) to (x_2, y_2). Show that

$$\int_C (-y\,dx + x\,dy) = x_1y_2 - x_2y_1. \qquad [9]$$

Let (x_1, y_1), (x_2, y_2), ... , (x_n, y_n) be the vertices of a polygon, in anticlockwise order. Show that the area of the polygon is

$$\tfrac{1}{2}\left[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + \cdots + (x_{n-1}y_n - x_ny_{n-1}) + (x_ny_1 - x_1y_n)\right]. \qquad [7]$$

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