UNIVERSITY OF SURREY©

B. Sc. Undergraduate Programmes in Mathematical Studies
M. Math. Undergraduate Programmes in Mathematical Studies

Level HE1 Examination

Module MS102 Vector Calculus

Time allowed – 2 hrs Spring Semester 2006

Attempt FOUR questions. If any candidate attempts more than FOUR questions only the best FOUR solutions will be taken into account.

Question 1

(a) Let $\mathbf{a} = (5, -3, -2)$ and $\mathbf{b} = (-3, 2, 6)$. Find $\mathbf{a} \cdot \mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$. [7]

(b) Find the two values of A such that the planes $Ax - y + z = 1$ and $3Ax + Ay - 2z = 6$ are perpendicular. [7]

(c) Explain why the two planes $2x - y + 5z = 5$ and $2x - y + 5z = 15$ are parallel. [2]

Find the equation of the straight line that is perpendicular to these planes and which passes through the point $(0, 0, 1)$ on the first plane. Find where this line intersects the second plane and hence show that the distance between the planes is $\frac{1}{3}\sqrt{30}$. [9]

Question 2

(a) Explain what it means for two straight lines in three dimensions to be *skew*. [2]

(b) In three dimensional space, two skew lines have the equations $\mathbf{r} = \mathbf{d}_1 + \lambda\,\mathbf{n}_1$ and $\mathbf{r} = \mathbf{d}_2 + \mu\,\mathbf{n}_2$. Let the point A on the first line, and the point B on the second, be such that the line joining A and B is perpendicular to both lines. By considering a certain round trip from the origin back to the origin and involving the two points A and B, or otherwise, show that the distance between the two skew lines is

$$\left| (\mathbf{d}_2 - \mathbf{d}_1) \cdot \left(\frac{\mathbf{n}_1 \times \mathbf{n}_2}{|\mathbf{n}_1 \times \mathbf{n}_2|} \right) \right|$$ [13]

(c) Find the distance between the skew lines $\mathbf{r} = (1, 1, 0) + \lambda(1, -1, 1)$ and $\mathbf{r} = (0, 1, 2) + \mu(2, 1, -1)$. [10]

Question 3

(a) A closed rectangular box is to have a volume of $2\,\mathrm{m}^3$. The top and bottom are made from material costing 10 pence per square metre, and the sides from material costing 5 pence per square metre. Show that the cost C of the box (in pence) is given by

$$C(x, y) = 20\left(xy + \frac{1}{x} + \frac{1}{y} \right)$$ [7]

where x and y are the length and breadth of the base. Find the dimensions of the box which minimise the cost of manufacture. [You must confirm that you have a minimum]. [11]

(b) Let $z = f(u, v)$ with $u = x + y$ and $v = x - y$. Show that

$$\left(\frac{\partial z}{\partial u}\right)^2 - \left(\frac{\partial z}{\partial v}\right)^2 = \frac{\partial z}{\partial x}\frac{\partial z}{\partial y}$$ [7]

Question 4

(a) Evaluate the integral $\iint_D (x^2 + y^2)\, dA$, where D is the triangular region with vertices $(0,0)$, $(1,0)$ and $(0,1)$. [10]

(b) Show that the sphere $x^2 + y^2 + z^2 = 2$ and the paraboloid $z = x^2 + y^2$ intersect in the circle $x^2 + y^2 = 1$, $z = 1$. [5]

(c) Using cylindrical polar coordinates, find the volume of the solid that is bounded above by the plane $z = 1$ and below by the paraboloid $z = x^2 + y^2$. [You may quote without proof that the Jacobian of the transformation into cylindrical polars is r]. [10]

Question 5

(a) Explain how to find the extrema of a function $f(\mathbf{x})$ subject to a side constraint of the form $g(\mathbf{x}) = 0$, where $\mathbf{x} \in \mathbb{R}^n$. [5]

(b) Let R be a given fixed number. Find the maximum value of

$$f(x, y, z) = \ln x + 2 \ln y + 3 \ln z$$

subject to the constraint that $x^2 + y^2 + z^2 = 6R^2$. [14]

Deduce that if x, y and z are positive real numbers then

$$xy^2z^3 \le 6\sqrt{3} \left(\frac{x^2 + y^2 + z^2}{6} \right)^3 . \qquad [6]$$

Question 6

(a) State Green's theorem and use it to show that if a region in the (x, y) plane is bounded by a closed curve C then the area of the region is

$$\tfrac{1}{2} \int_C (-y\, dx + x\, dy) \qquad [9]$$

where the curve is traversed anticlockwise. Use this result to find the area of the ellipse $\frac{1}{9}x^2 + \frac{1}{4}y^2 = 1$. [7]

(b) Suppose that C is the line segment from the point $(1, 1)$ to $(3, 2)$. Evaluate

$$\tfrac{1}{2} \int_C (-y\, dx + x\, dy)$$

and state, with reasons, whether the integral represents the area of anything in this case. [9]

INTERNAL EXAMINER: S.A. Gourley