Unassessed Coursework # 2

1. Let $(\mathbf{x}, U)$ be a coordinate chart for a surface $\mathcal{M} \subset \mathbb{R}^3$ with $U = \mathbb{R}^2$. Give a condition for the chart to be *regular*. Determine all values of $(u, v) \in \mathbb{R}^2$ where the following coordinate chart is *not regular*: $\mathbf{x}(u, v) = \big(u^2 + v^2, u + v, v^2 + u^2\big)$.

2. Let $(\mathbf{x}, U)$ be a regular coordinate chart for a surface $\mathcal{M} \subset \mathbb{R}^3$. Define the following curvatures: (a) the principal curvatures κ_1 and κ_2, (b) Gaussian curvature $\mathcal{K}$, and (c) mean curvature $\mathcal{H}$.

3. A point p on a surface $\mathcal{M} \subset \mathbb{R}^3$ is called

   - elliptic if $\mathcal{K} > 0$,
   - hyperbolic if $\mathcal{K} < 0$,
   - parabolic if $\mathcal{K} = 0$ and $\mathcal{H} \neq 0$,
   - planar if $\mathcal{K} = \mathcal{H} = 0$,

   where $\mathcal{K}$ is the Gaussian curvature and $\mathcal{H}$ is the mean curvature. Consider the coordinate chart

$$\mathbf{x}(u, v) = \begin{pmatrix} u + v \\ u - v \\ uv \end{pmatrix},$$

   for $u, v \in \mathbb{R}^2$. Compute the Gaussian $\mathcal{K}$ and mean curvature $\mathcal{H}$ at the point $u = v = 1$ and determine whether this point on $\mathcal{M}$ is elliptic, hyperbolic or parabolic.

4. Let $(\mathbf{x}, U)$ be a coordinate chart for a surface $\mathcal{M} \subset \mathbb{R}^3$ defined by

$$\mathbf{x}(u, v) = \begin{pmatrix} u \\ v \\ u^2 + v^2 \end{pmatrix}, \quad (u, v) \in \mathbb{R}^2 .$$

   Compute the coefficients E, F and G in the first fundamental form.

5. Let $(\mathbf{x}, U)$ be a coordinate chart for a surface $\mathcal{M} \subset \mathbb{R}^3$ defined by

$$\mathbf{x}(u, v) = \begin{pmatrix} u \\ f(u, v) \\ v \end{pmatrix}, \quad (u, v) \in \mathbb{R}^2 ,$$

   where $f(u, v)$ is a given smooth function. Show that the Gaussian curvature is

$$\mathcal{K} = \frac{\big(f_{uu} f_{vv} - f_{uv}^2\big)}{(1 + f_u^2 + f_v^2)^2} .$$

6. Show that

$$\mathbf{n}_u \times \mathbf{n}_v = \mathcal{K}\sqrt{EG - F^2}\,\mathbf{n}\,,$$

where $\mathcal{K}$ is the Gaussian curvature and $\mathbf{n}$ is the usual unit normal at the surface.

7. Let $\boldsymbol{\gamma}(u)$ be a regular unit-speed curve in $\mathbb{R}^3$ with $\kappa(u) > 0$, and let $\mathbf{t}(u) = \boldsymbol{\gamma}'(u)$. Construct a surface based on the chart

$$\mathbf{x}(u,v) = \boldsymbol{\gamma}(u) + v\mathbf{t}(u)\,, \quad v > 0 \text{ and } u \in \mathbb{R}\,.$$

Show that the coefficients of the first fundamental form are $E = 1 + v^2\kappa^2$, $F = 1$ and $G = 1$ where $\kappa(u)$ is the curvature of the curve $\boldsymbol{\gamma}(u)$. Show that

$$N = \mathbf{n} \cdot \mathbf{x}_{vv} = 0\,,$$

and conclude that $\mathcal{K} \leq 0$ at all points on the surface.

8. In the previous question it is established that every point on the surface is either hyperbolic, parabolic or planar. In this question the precise form of the surface curvature is resolved. First show that the normal vector to the surface is

$$\mathbf{n} = \frac{1}{\kappa}\mathbf{t}'(u) \times \mathbf{t}(u)\,.$$

Use this vector to compute

$$M = \mathbf{n} \cdot \mathbf{x}_{uv}\,,$$

and show that $\mathcal{K} = 0$ everywhere on the surface. By computing $L = \mathbf{n} \cdot \mathbf{x}_{uu}$ as well, show that a point on the surface is planar if $\tau(u) = 0$ and parabolic if $\tau(u) \neq 0$, where $\tau(u)$ is the torsion of the curve $\boldsymbol{\gamma}(u)$.

9. Let $\mathcal{M} \subset \mathbb{R}^3$ be a surface with coordinate chart $(\mathbf{x}, U)$ where

$$\mathbf{x}(u,v) = \begin{pmatrix} h(u)\cos v \\ u \\ h(u)\sin v \end{pmatrix}\,,$$

and

$$U = \{\,(u,v) \in \mathbb{R}^2 \ :\ -\infty < u < +\infty\,,\ 0 \leq v < 2\pi\,\}\,.$$

The function $h(u)$ is a given smooth function with: $h(u) > 0$ and $h'(u) > 0$.

(a) Determine the coefficients of the first fundamental form, E, F, G, for this surface and show that the chart is regular.

(b) Determine the coefficients of the second fundamental form, L, M, N, for this surface.

(c) Determine the differential equation that $h(u)$ has to satisfy if the *mean curvature* $\mathcal{H}$ is required to vanish at all points on the surface. (You don't need to solve this differential equation.)

(d) Suppose that the mean curvature vanishes everywhere on the surface $\mathcal{M}$ and so $h(u)$ satisfies the differential equation in part (c). Use the differential equation to show that the *Gaussian curvature* $\mathcal{K}$ is strictly negative at each point on the surface.