MAT2047: Curves and Surfaces Spring 2012

Unassessed Coursework # 1

1. Compute the curvature $\kappa(t)$ and torsion $\tau(t)$ for the space curve

$$\boldsymbol{\gamma}(t) = \begin{pmatrix} 4\cos t \\ 5 - 5\sin t \\ -3\cos t \end{pmatrix} ,$$

where $0 \le t < 2\pi$.

2. Show that the space curve with parameterisation

$$\boldsymbol{\gamma}(t) = \begin{pmatrix} 5\cos t \\ 3\cos t - 4\sin t \\ 4\cos t + 3\sin t \end{pmatrix} , \quad 0 \le t < 2\pi ,$$

is planar. That is, show that there exists a constant vector $\mathbf{m} \in \mathbb{R}^3$ such that

$$\mathbf{m} \cdot \dot{\boldsymbol{\gamma}}(t) = 0 ,$$

for all $t \in [0, 2\pi)$.

3. The intersection of the two surfaces in $\mathbb{R}^3$, defined implicitly by

$$x^2 + y^2 = 1 \quad \text{and} \quad x + y + z = 1 ,$$

is a curve. Find a regular parameterisation for this curve.

4. Consider the parametrised curve

$$\boldsymbol{\gamma}(t) = \begin{pmatrix} 2t \\ t^2 \\ \ln(t) \end{pmatrix} , \quad 0 < t < +\infty .$$

Show that this curve passes through the points

$$\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 4 \\ 4 \\ \ln(2) \end{pmatrix} ,$$

and find the arc-length of the curve between these two points.

5. Show that

$$\boldsymbol{\gamma}(t) = \begin{pmatrix} t \\ \sin t \\ e^t \end{pmatrix} , \quad -\infty < t < +\infty ,$$

and

$$\widetilde{\boldsymbol{\gamma}}(\widetilde{t}) = \begin{pmatrix} \ln(\widetilde{t}) \\ \sin\left(\ln(\widetilde{t})\right) \\ \widetilde{t} \end{pmatrix} , \quad 0 < \widetilde{t} < +\infty ,$$

both represent the same curve. That is, show that $\widetilde{\boldsymbol{\gamma}}(\widetilde{t})$ is a *reparametrisation* of $\boldsymbol{\gamma}(t)$.

6. Let $\boldsymbol{\gamma}(s)$ be a unit-speed curve in $\mathbb{R}^3$, and assume that its curvature $\kappa(s)$ is non-zero for all s. Define a new curve by

$$\boldsymbol{\delta}(s) = \boldsymbol{\gamma}'(s)\,.$$

Show that $\boldsymbol{\delta}(s)$ is regular but that it is not in general unit speed, by showing that

$$\frac{d}{ds}\mathcal{S} = \kappa(s)\,, \quad \mathcal{S}(s) := \int_0^s \|\boldsymbol{\delta}'(u)\|\,\mathrm{d}u\,,$$

that is, $\mathcal{S}$ is the arclength of the curve $\boldsymbol{\delta}(s)$. Show that the curvature of $\boldsymbol{\delta}(s)$ is

$$\mathcal{K}(s) = \left(1 + \frac{\tau^2}{\kappa^2}\right)^{1/2}\,,$$

when $\kappa(s)$ and $\tau(s)$ are the curvature and torsion respectively for the curve $\boldsymbol{\gamma}(s)$.

7. Let $\kappa(t)$ be the curvature of a regular parameterised curve $\boldsymbol{\gamma}(t) = (\gamma_1(t), \gamma_2(t))$ in the plane. Let $\mathbf{R}$ be a rotation matrix. Every rotation matrix in the plane is of one of two forms

$$\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{or} \quad \mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & -\cos\theta \end{bmatrix}\,,$$

where $0 \le \theta \le 2\pi$. Let

$$\widetilde{\boldsymbol{\gamma}}(t) = \mathbf{R}\boldsymbol{\gamma}(t)\,,$$

be the original curve rotated in the plane using $\mathbf{R}$. Show that the curvature $\widetilde{\kappa}(t)$ of the rotated curve $\widetilde{\boldsymbol{\gamma}}$ satisfies

$$\widetilde{\kappa}(t) = \det(\mathbf{R})\kappa(t)\,.$$

8. A curve $\boldsymbol{\gamma}(t)$ is said to be a *spherical curve* if there exists a constant vector $\mathbf{a} \in \mathbb{R}^3$ (the centre) and a constant radius $r > 0$ such that

$$(\boldsymbol{\gamma}(t) - \mathbf{a}) \cdot (\boldsymbol{\gamma}(t) - \mathbf{a}) = r^2\,, \quad \text{for all } t\,. \tag{1}$$

Suppose $\kappa(t) \neq 0$ and $\tau(t) \neq 0$ for this curve. Show that they satisfy

$$\frac{1}{\kappa^2} + \left(\frac{\dot{\kappa}}{\ell\tau\kappa^2}\right)^2 = r^2 \quad \text{for all} \quad t\,, \quad \text{where } \dot{\kappa} = \frac{d\kappa}{dt}\,.$$

(Hint: express $\boldsymbol{\gamma}(t) - \mathbf{a} = c_1\mathbf{T} + c_2\mathbf{N} + c_3\mathbf{B}$, where $(\mathbf{T}, \mathbf{N}, \mathbf{B})$ is a Frenet-Serret frame, and determine the coefficients c_1, c_2 and c_3 by differenting equation (1) (several times).)

Solutions will be posted for downloading at

`http://personal.maths.surrey.ac.uk/st/T.Bridges/MODULES/MAT2047/`