UNIVERSITY OF SURREY©

Faculty of Engineering & Physical Sciences

Department of Mathematics

Undergraduate Programmes in Mathematical Studies

Module MAT2044; 30 Credits

Applied II: Linear Partial Differential Equations

Level HE2 Examination

Time allowed: Two hours

Semester 2, 2009/10

Attempt THREE questions
If a candidate attempts more than THREE questions only the best THREE questions will be taken into account.

Each question carries 25 marks.

Where appropriate the mark carried by an individual part of a question is indicated in square brackets [ ].

Approved calculators are allowed.

Additional material:
None

Question 1

(a) State the *mean value property* for harmonic functions. [2]

(b) Let D be a disk in $\mathbb{R}^2$. Suppose that u_1 and u_2 satisfy

$$\triangle u_j = u_j \quad \text{for } x \in D$$

for $j = 1, 2$ subject to

$$u_1|_{\partial D} = u_2|_{\partial D} = g,$$

where g is a given continuous function.

(i) Show that $w = u_1 - u_2$ is zero on ∂D and write down the PDE satisfied by w. [2]

(ii) By multiplying this PDE by w and integrating over D, prove that w must be identically zero in D. The identity

$$\int_D w \triangle w \, dx \, dy = -\int_D |\nabla w|^2 \, dx \, dy$$

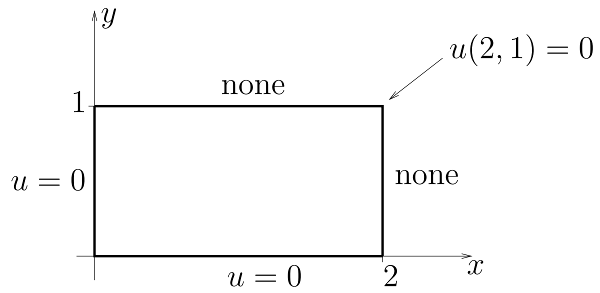
may be used without proof. [3]

(iii) How do parts (b)(i) and (b)(ii) prove that the solution to $\triangle u = u$ in D subject to $u = g$ on ∂D is unique? [2]

(c) Consider $\triangle u = 0$ on the rectangle R given by

$$R = \{(x, y) : \ 0 < x < 2, \ 0 < y < 1\}$$

with boundary conditions as indicated here:



(i) By separating the variables, show that the general solution is of the form

$$u(x, y) = \sum_{n=1}^{\infty} A_n \sinh(n\pi x) \sin(n\pi y) + B_n \sinh\left(\frac{n\pi y}{2}\right) \sin\left(\frac{n\pi x}{2}\right),$$

where the A_n and B_n are constants. (You may assume that the Laplace operator in cartesian coordinates is given by $\triangle u = u_{xx} + u_{yy}$.) [12]

(ii) Suppose we also wish to impose the boundary condition $u(x, 1) = h(x)$, where h is a given continuous function satisfying $h(0) = h(2) = 0$. Derive a formula for the B_n in terms of h. [4]

Question 2

(a) (i) Compute the Fourier series of the function

$$f(x) = \begin{cases} 3 & \text{if } -1 < x < 0 \\ 0 & \text{if } 0 < x < 1 \end{cases}$$

extended with period 2 to the real line. [5]

(ii) What can you say about the convergence of the Fourier series? What is its limit? [4]

(b) Recall that the Fourier transform of a rapidly decaying function f is given by

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx.$$

Prove that the Fourier transform of the function $f_{bt}(x) := f(x + bt)$ satisfies

$$\widehat{f_{bt}}(k) = e^{ikbt} \hat{f}(k),$$

where b and t are constants. [4]

(c) Show that the Fourier transform of the solution of

$$\begin{aligned} u_t &= u_{xx} + bu_x \qquad x \in \mathbb{R},\ t > 0 \\ u(x,0) &= u_0(x), \end{aligned} \tag{1}$$

where u_0 is a given continuous and rapidly decaying function and b is a constant, satisfies

$$\hat{u}(k,t) = e^{ikbt} e^{-k^2 t} \hat{u_0}(k).$$

You may use without proof the fact that

$$\widehat{\frac{d^j f}{dx^j}}(k) = (ik)^j \hat{f}(k)$$

for any positive integer j. [5]

(d) Suppose v solves the heat equation

$$\begin{aligned} v_t &= v_{xx} \qquad x \in \mathbb{R}, \quad t > 0 \\ v(x,0) &= u_0(x). \end{aligned}$$

What equation is satisfied by the Fourier transform $\hat{v}(k,t)$ of v? [2]

(e) Now assume that the solution v to the equation in part (d) is given by

$$v(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{(x-y)^2}{4t}} u_0(y) \mathrm{d}y.$$

Using this and your answers to parts (b),(c) and (d) above, or otherwise, write down a formula for the solution u of equation (1). [5]

Question 3

(a) (i) Find the general solution of

$$u_y + (\cos x)^2 u_x = 0 \qquad (2)$$

and plot the characteristics. [7]

(ii) Find a solution of (2) that satisfies $u(x,1) = (1 - \tan x)^2$ for $0 < x < \frac{\pi}{2}$. [3]

(iii) Does data prescribed on the line $\{(\frac{\pi}{2}, y) : y > 0\}$ determine a solution of (2)? Justify your answer. [2]

(b) Consider the equation

$$3u_{xx} - u_{xt} - 2u_{tt} = 0. \qquad (3)$$

(i) Let $\xi = x + at$ and $\eta = x + bt$. Use the chain rule to write (3) in terms of ξ and η.

(ii) Choose the constants a and b so that the transformed PDE is

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = 0.$$

(iii) Hence show that the general solution to (3) is

$$u(x,t) = F(2x - 3t) + G(x + t).$$

[7]

(c) Let u solve the transport equation

$$cu_x + u_t = 0 \quad x \in \mathbb{R},\ t > 0$$

subject to $u(x,0) = u_0(x)$, where u_0 is a given differentiable function defined on $\mathbb{R}$.

(i) Assuming c is constant, find a solution to this problem. [2]

(ii) Now assume that c is a function of time $c(t) = \dot{k}(t)$ for some smooth function k. Find the characteristics of the transport equation under these assumptions. Hence write down the general solution. [4]

Question 4

(a) Using separation of variables, show that the general solution of

$$\begin{aligned} u_t &= u_{xx} + u \qquad 0 < x < \pi,\ t > 0 \\ u_x(0,t) &= u_x(\pi,t) = 0 \quad t > 0 \end{aligned}$$

is

$$u(x,t) = A_0 e^t + \sum_{n=1}^{\infty} A_n e^{(1-n^2)t} \cos(nx),$$

where the A_n are constants.

If $u(0,t)$ is to remain bounded for all time, what value should A_0 take? [9]

(b) Let u be a solution of the following problem:

$$\begin{aligned} u_t &= u_{xx}, & 0 < x < 1, t > 0 \\ u(0,t) &= \frac{1}{4}t^2, & t \geq 0 \\ u(1,t) &= \frac{1}{8}t^2, & t \geq 0 \\ u(x,0) &= 4x(1-x), & 0 \leq x \leq 1. \end{aligned}$$

For each $T \geq 0$, let $\Omega(T) = \{(x,t) :\ 0 \leq x \leq 1,\ 0 \leq t \leq T\}$.

(i) State the weak maximum principle for solutions of the heat equation subject to the boundary conditions above. [3]

(ii) Using your answer to (b)(i), show that there is at most one solution to this PDE problem. [3]

(iii) What is the maximum and minimum value of u over the set $\Omega(2)$, and where in the set do they occur? [4]

(c) Consider the three figures overleaf, each containing a solution at time $t = 0$ (solid curve) and $t = 1$ (dotted curve).

(i) For each figure, state whether the solution could correspond to a solution of the wave equation subject to the initial condition $u_t(x,0) = 0$ for $x \in \mathbb{R}$. Justify your answer.

(ii) Could any of the three figures represent a solution to the heat equation? Justify your answer. [6]

INTERNAL EXAMINER: J. Bevan
EXTERNAL EXAMINER: D. Chillingworth

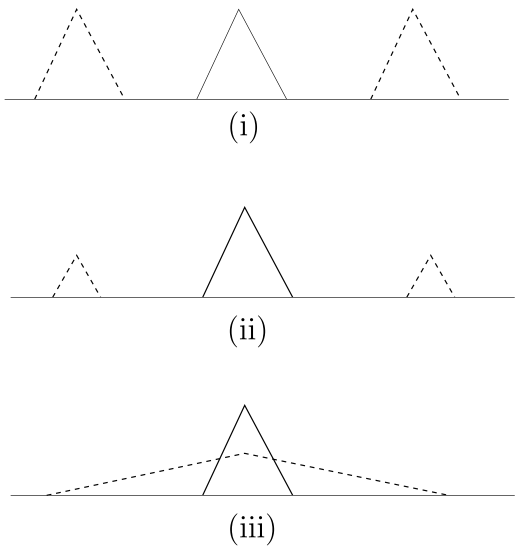


Figure 1: Figures for use in Question 4 (c).

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