UNIVERSITY OF SURREY©

Faculty of Engineering & Physical Sciences

Department of Mathematics

Undergraduate Programmes in Mathematical Studies

Module MAT2044; 30 Credits

Applied II: Fluid Dynamics

Level HE2 Examination

Time allowed: Two hours Semester 1, 2009/10

Answer ALL parts of the question in Section A.
Answer ONE question in Section B; if you answer both questions, the question with the lower marks will be discounted.

The question in Section A carries 60 marks; each question in Section B carries 40 marks.

Where appropriate the mark carried by an individual part of a question is indicated in square brackets [ ].

Approved calculators are allowed.

Additional material:
1 handout

Section A

Attempt ALL parts of the question in this section

Question 1

(a) State the principle of conservation of mass and use it to derive the continuity equation,

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{u} = 0.$$

(You may use Reynolds' Transport Theorem without proving it.) [9]

(b) Explain the meaning of each of the following terms in italics. (A clear explanation is needed; do not merely reproduce the formula for any of these terms.)

(i) A *two-dimensional* flow.

(ii) The *density* of a fluid.

(iii) The *stress* at a point on the surface of a fluid blob. [9]

(c) Use Cartesian coordinates to prove the identity

$$\nabla\times(\nabla\times\mathbf{F}) = \nabla(\nabla\cdot\mathbf{F}) - \nabla^2\mathbf{F}.$$ [8]

(d) Describe two examples of *instability* of a steady viscous flow, using sketches to illustrate each example. [8]

(e) Working in Cartesian coordinates, use the Divergence Theorem to prove that, for any vector field $\mathbf{F}(\mathbf{x}) = f(\mathbf{x})\mathbf{e}_x + g(\mathbf{x})\mathbf{e}_y + h(\mathbf{x})\mathbf{e}_z$, the following identity holds:

$$\int_V \nabla\times\mathbf{F}\,\mathrm{d}\mathbf{x} = \int_{\partial V}\mathbf{n}\times\mathbf{F}\,\mathrm{d}S.$$ [9]

(f) Consider the following unsteady unidirectional two-dimensional incompressible viscous flow in a channel $0 \le y \le a$. The fluid is stationary until $t = 0$, at which time the solid boundary $y = a$ begins moving with velocity $U\mathbf{e}_x$, where $U > 0$. The solid boundary $y = 0$ remains stationary always. Describe how the velocity profile changes with time, illustrating your explanation with at least three sketches showing the velocity profile at different stages after the flow starts. You may assume that no instability occurs. (Do not try to solve the Navier-Stokes equations – only a description of the flow is needed.) [9]

(g) A two-dimensional incompressible viscous flow, $\mathbf{u} = u(x,y,t)\mathbf{e}_x + v(x,y,t)\mathbf{e}_y$ (in Cartesian coordinates), can be described in terms of a streamfunction $\psi(x,y,t)$ by the relations $u = \psi_{,y}$, $v = -\psi_{,x}$. Show that

$$(\nabla^2\psi)_{,t} + \psi_{,y}(\nabla^2\psi)_{,x} - \psi_{,x}(\nabla^2\psi)_{,y} = \nu\nabla^2(\nabla^2\psi).$$ [8]

Section B

Attempt ONE question from this section

Question 2

(a) State a sufficient condition for an irrotational two-dimensional flow to be a potential flow, i. e. $\mathbf{u} = \nabla\varphi$. Explain the meaning of any technical terms that you use. [7]

(b) For a steady two-dimensional inviscid potential flow (in Cartesian coordinates), explain why the potential $\varphi(x, y)$ and the streamfunction $\psi(x, y)$ satisfy the Cauchy-Riemann equations:

$$\varphi_{,x} = \psi_{,y}\,, \qquad \varphi_{,y} = -\psi_{,x}\,.$$

Use this result to prove that the complex potential $\Phi = \varphi + i\psi$ is a function of $\zeta = x + iy$ only. [8]

(c) Use cylindrical polar coordinates to calculate the inviscid flow $\mathbf{u} = \varphi_{,r}\,\mathbf{e_r} + \frac{1}{r}\,\varphi_{,\theta}\,\mathbf{e}_\theta$ that corresponds to the complex potential $\Phi = (a - ib)\ln\zeta$, where a and b are real constants. Sketch the streamlines, adding arrows to show the direction of flow, for

(i) $a = 0$ and $b > 0$;

(ii) $a < 0$ and $b = 0$;

(iii) $a < 0$ and $b > 0$.

[17]

(d) Compare the flow in part (c)(iii) with a ‘bathtub vortex,’ which is formed when water drains from a bath and rotates around the plughole. What qualitative features do the two flows have in common? How do they differ? [8]

Question 3

(a) Let $G(\mathbf{x},t)$ be a differentiable function that represents a property of a fluid and let $V(t)$ denote the bounded region of space that is occupied by a given blob of fluid at time t. Prove Reynolds' Transport Theorem,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} G(\mathbf{x},t)\,\mathrm{d}\mathbf{x} = \int_{V(t)} \left\{ \frac{DG(\mathbf{x},t)}{Dt} + G(\mathbf{x},t)\nabla\cdot\mathbf{u} \right\} \mathrm{d}\mathbf{x}.$$

Hint: Consider the region $V(0)$ that the blob occupied at time 0, so that the fluid particle which occupies position $\mathbf{x}\in V(t)$ at time t occupied the position $\mathbf{X}\in V(0)$ at time 0. You may use without proof the result that the Jacobian determinant J, such that $\mathrm{d}\mathbf{x} = J(\mathbf{X},t)\,\mathrm{d}\mathbf{X}$, satisfies

$$\frac{\partial}{\partial t}\big(J(\mathbf{X},t)\big) = (\nabla\cdot\mathbf{u})J(\mathbf{X},t).$$ [9]

(b) Use Reynolds' Transport Theorem and the continuity equation to show that for an arbitrary differentiable function $F(\mathbf{x},t)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \rho(\mathbf{x},t)F(\mathbf{x},t)\,\mathrm{d}\mathbf{x} = \int_{V(t)} \rho(\mathbf{x},t)\frac{DF(\mathbf{x},t)}{Dt}\,\mathrm{d}\mathbf{x}.$$ [6]

(c) Use Cartesian coordinates to prove the following identity for $\mathbf{u} = u\mathbf{e}_x + v\mathbf{e}_y + w\mathbf{e}_z$:

$$\nabla^2\left(\tfrac{1}{2}\,\mathbf{u}\cdot\mathbf{u}\right) = \mathbf{u}\cdot\left(\nabla^2\mathbf{u}\right) + |\nabla u|^2 + |\nabla v|^2 + |\nabla w|^2.$$ [10]

(d) The total *kinetic energy* of the fluid in the blob at time t is

$$E(t) = \int_{V(t)} \frac{\rho}{2}\,(\mathbf{u}\cdot\mathbf{u})\,\mathrm{d}\mathbf{x}.$$

Now suppose that the fluid is incompressible, that it has viscosity $\mu > 0$, and that $V(t)$ is a fixed volume V whose boundary ∂V is solid and stationary. Show that

$$\frac{\mathrm{d}E(t)}{\mathrm{d}t} \leq 0.$$

(Use vector notation rather than any particular coordinate system.) [15]

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