Faculty of Engineering & Physical Sciences

Department of Mathematics

Undergraduate Programmes in Mathematical Studies

Module MAT2043; 30 Credits

Applied I
Numerical and Computational Methods

Level HE2 Examination

Time allowed: $1\frac{1}{2}$ hours

Semester 2, 2009/10

Answer **THREE** questions only
If a candidate attempts more than THREE questions,
then only the best THREE solutions will be taken into account.

Each question carries 20 marks.

Where appropriate the mark carried by an individual
part of a question is indicated in square brackets [ ].

Approved calculators are allowed.

Additional material:
None

Question 1

(a) The Jacobi, Gauss-Seidel and SOR iterations for solving the linear system $A\mathbf{x} = \mathbf{b}$ can all be written in the form

$$\mathbf{x}^{(k+1)} = M\mathbf{x}^{(k)} + \mathbf{c}. \qquad (1)$$

(i) By splitting the matrix A into diagonal, lower triangular and upper triangular parts as $A = D + L + U$, derive the Jacobi iteration in matrix form. [3]

(ii) State a condition on M for the iteration (1) to converge. [1]

(iii) The solution $\mathbf{x}$ of the linear system is also a fixed point of the iteration (1). Show that the errors $\mathbf{e}^{(k)} = \mathbf{x} - \mathbf{x}^{(k)}$ satisfy

$$\mathbf{e}^{(k+1)} = M\mathbf{e}^{(k)}.$$ [2]

(iv) Derive the error bound

$$||\mathbf{e}^{(k)}|| \leq ||M||^k ||\mathbf{e}^{(0)}||.$$

[Hint: You may use the results that $||A\mathbf{x}|| \leq ||A||\,||\mathbf{x}||$ and $||AB|| \leq ||A||\,||B||$.] [4]

(b) (i) Consider the iteration

$$x_{n+1} = g(x_n).$$

Suppose that $x_n \to \alpha$ as $n \to \infty$ for a given initial value x_0.
State the definition of linear and quadratic convergence of this sequence.
State conditions on g for the iteration to converge linearly. [3]

(ii) State the modified Newton method for solving the single nonlinear equation $f(x) = 0$ in the form

$$x_{n+1} = g_m(x_n). \qquad (2)$$

When would it be appropriate to use this iteration rather than the Newton iteration? [3]

(iii) Suppose that the modified Newton iteration (2) converges to the fixed point α for a given initial value x_0. Find $g_m'(\alpha)$. State conditions that must be satisfied for the modified Newton iteration to converge linearly. [4]

Question 2

(a) Derive the Lagrange form of the interpolating polynomial of degree n which satisfies

$$p_n(x_i) = f_i, \qquad i = 0, 1, \ldots, n,$$

where f_i is the function value at x_i, $i = 0, 1, \ldots, n$. [10]

(b) Show that the interpolating polynomial of degree n is unique. [5]

(c) Find the quadratic polynomial which interpolates the following data and hence find an approximation to $f(1)$.

$$x_0 = 0, \quad x_1 = 2, \quad x_2 = 4, \qquad f_0 = 2, \quad f_1 = 4, \quad f_2 = -2.$$ [5]

Question 3

(a) (i) Prove the result

$$f'(x_0) = \frac{f(x_0+h) - f(x_0-h)}{2h} + ch^2 + O(h^4)$$

and find the value of the constant c, which is independent of h. [4]

(ii) Using this result, and the corresponding result obtained by replacing h by $h/2$, derive an $O(h^4)$ approximation to $f'(x_0)$. [5]

(b) Let $p_n(x)$ be the polynomial of degree n which interpolates the function $f(x)$ at the equally spaced points x_k, $k = 0, 1, \ldots, n$, where $x_0 = a$ and $x_n = b$. Then you may assume that

$$f(x) = p_n(x) + e_n(x)$$

where

$$p_n(x) = \sum_{k=0}^{n} f(x_k) L_{n,k}(x), \qquad e_n(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \prod_{i=0}^{n} (x - x_i)$$

with $\xi(x) \in (a, b)$.

(i) Use this result to derive the Newton-Cotes formulae for approximating $I(f) = \int_a^b f(x)\,dx$, including the error term. [4]

(ii) Derive the Trapezoidal Rule ($n = 1$). [3]

(iii) State the error term for the case $n = 1$ and simplify it using the Mean Value Theorem for Integrals. [4]

Question 4

(a) (i) Apply the Backward Euler Method to the differential equation

$$\dot{y} = -2(t+1)y + \sin t, \qquad y(0) = 1$$

and rearrange the iteration to make it explicit. [4]

(ii) The damped pendulum equation is given by

$$\ddot{y} + \delta\dot{y} + \omega^2 \sin y = 0, \qquad y(0) = \alpha, \quad \dot{y}(0) = \beta.$$

Write this second order equation as two first order equations and state the iteration obtained by applying Euler's Method to this system of equations. [5]

(b) Consider the linear boundary value problem

$$y'' - 3y' + y = 0,$$

$$y(0) = 1, \qquad y'(1) = 2.$$

Apply the finite difference method to this problem using the mesh points $x_i = ih$ with $h = 0.25$. Replace the first derivative term in the differential equation and the boundary condition with an $O(h^2)$ approximation. Hence set up, but do *not* attempt to solve, a linear system of equations for the determination of $y_i \simeq y(x_i)$. [11]

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