MAT2011 Spring 2011: Unassessed Assignment 7.8

7.8.1 Using d'Alembert's method, solve the wave equation $c^2 u_{xx} = u_{tt}$ for $x \in \mathbb{R}$ and $t > 0$ subject to

$$\begin{aligned} u(x,0) &= e^{-x^2} \\ u_t(x,0) &= (\cos x)^2 \end{aligned}$$

for $x \in \mathbb{R}$.
If the initial speed $u_t(x,0)$ had been idenitcally zero, sketch the solution for times $t = 0$, $t = 1$ and $t = 100$.

7.8.2 A flea is sitting on a piece of string at a distance M from the origin. A hammer of width a strikes the string on the interval $[0,a]$. By modelling the disturbance caused in the string as a solution to the wave equation $c^2 u_{xx} = u_{tt}$, calculate when the flea first feels the effect of the hammer strike.

7.8.3

(i) Show that the equation

$$3u_{xx} - 2u_{xt} - u_{tt} = 0$$

is hyperbolic and find its general solution. Sketch the characeristics in the (x,t) plane.

(ii) Consider the following non-homogeneous version of the equation above:

$$3u_{xx} - 2u_{xt} - u_{tt} = 4(u_x - u_t).$$

By changing to the new variables

$$\begin{aligned} \xi &= x - 3t \\ \eta &= x + t, \end{aligned}$$

show that the transformed version of the equation is

$$u_{\xi\eta} = u_\xi.$$

Hence find the general solution of this equation.

7.8.4

(i) Fourier transform both sides of the equation

$$-u_t = u_{xx}$$

and both sides of the initial condition

$$u(x,0) = u_0(x)$$

in order to derive an ODE for $\hat{u}(k,t)$. Solve the resulting ODE (as we did in lectures, although beware the change of sign).

(ii) Can you apply the inverse Fourier transform to the expression you obtain for $\hat{u}(k,t)$? Justify your answer.

(iii) What does this tell you about solutions of the heat equation 'backwards in time?'

[Hint for (iii): $U(x,t) = u(x,-t)$ solves the heat equation 'forwards in time'.]

7.8.5 Let u solve the equation

$$u_t = u_{xx} - au \quad 0 < x < l,\ t > 0$$

subject to $u_x(0,t) = u_x(l,t) = 0$ for all t. Here, a is a real parameter.

(i) Show that $V(t)$ defined by

$$V(t) = \frac{1}{2}\int_0^l u^2(x,t)\,dx.$$

satisfies

$$\dot{V}(t) = -\int_0^l (u_x^2 + au^2)\,dx.$$

(ii) Deduce that

$$\dot{V}(t) \leq -2aV(t)$$

for all t.

(iii) Let $a > 0$. Show that if $\dot{V}(t_0) = 0$ then $V(t_0) = 0$, and hence that $V(t) = 0$ for all $t \geq t_0$.

(iv) The result of (iii) above implies that V is strictly decreasing. Does this tell you anything about the limit $\lim_{t\to\infty} V(t)$?

(v) By integrating the inequality in (ii), show that in fact $V(t) \to 0$ as $t \to \infty$. [You may assume $V(0) < \infty$.]

(vi) Now suppose $a \leq 0$. Show that $u = e^{-at}$ is a solution of the equation which does *not* converge to zero as $t \to \infty$.