MAT2011 Spring 2011: Supplementary Exercises involving Separation of Variables

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1. Find the general solution $u(x,t)$ of

$$\begin{aligned} u_t &= u_{xx} - u, && 0 < x < \ell, \quad t > 0 \\ u_x(0,t) &= 0 = u_x(\ell,t), && t > 0. \end{aligned}$$

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2. Consider the eigenvalue problem

$$\begin{aligned} u_{xx} &= \lambda u, \qquad 0 < x < \ell \\ u_x(0) &= k_0 u(0) \\ u_x(\ell) &= -k_\ell u(\ell) \end{aligned}$$

with mixed boundary conditions, where k_0 and k_ℓ are given positive numbers. Can this system have a nontrivial solution $u \not\equiv 0$ for $\lambda > 0$?

Hint: Multiply the first equation by u and integrate over $x \in [0, \ell]$.

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3. Using the method of separation of variables, show that a possible general solution to

$$\begin{aligned} u_{xx} &= u_t, \qquad 0 < x < \ell,\ t > 0, \\ u_x(0) &= 0 \\ u_x(l) &= 0 \end{aligned}$$

is

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n e^{-\frac{n^2\pi^2 t}{l^2}} \cos\left(\frac{n\pi x}{l}\right),$$

where the A_n are real numbers. If the initial heat distribution satisfies

$$u(x,0) = (\cos\left(\frac{\pi x}{l}\right))^2$$

find the A_n. [The last part of the question can safely be left until Fourier series are introduced at the end of week 2.]
