MAT2011 Spring 2011: Unassessed Homework Assignment #5

5.1 Find a separable variables solution to Laplace's equation

$$\triangle u = 0$$

in the quarter disk

$$Q = \left\{(r,\theta):\ 0 \leq r < a,\ 0 \leq \theta \leq \frac{\pi}{2}\right\}$$

subject to

$$\begin{aligned} u(a\cos\theta, a\sin\theta) &= h(\theta) && 0 \leq \theta \leq \frac{\pi}{2} \\ u(x,0) &= 0 && 0 < x < a \\ u(0,y) &= 0 && 0 < y < a. \end{aligned}$$

[Hint - We have discussed the Laplacian in plane polar coordinates during lectures; you should adapt that method to the domain Q and use the principle of superposition.]

5.2 Suppose

$$\triangle u(x,y) = \quad (x,y) \in E,$$

where E is the ellipse

$$E = \{(ar\cos\theta, br\sin\theta):\ 0 \leq \theta \leq 2\pi,\ 0 \leq r \leq 1\}.$$

(Here, a and b are fixed positive constants.)

(i) If $u(x,y) = (x+y)^2$ whenever $(x,y) \in \partial E$, find the maximum and minimum value of u on the domain E.

(ii) Now suppose $b = a$. Find $u(0,0)$.

5.3 Suppose u_1 and u_2 solve Laplace's equation $\triangle u_i = 0$ on the unit disk in $\mathbb{R}^2$ subject to

$$\begin{aligned} u_1 &= g_1 \text{ if } (x,y) \in \partial D \\ u_2 &= g_2 \text{ if } (x,y) \in \partial D. \end{aligned}$$

Suppose further that $\min_{\partial D} g_1 > \max_{\partial D} g_2$ for all $(x,y) \in \partial D$. Prove that

$$u_1(x,y) > u_2(x,y)$$

for all points $(x,y) \in D$.

5.4 Let u_1 and u_2 be solutions of

$$\begin{aligned} u_t &= u_{xx} - u && 0 < x < 1,\ t > 0 \\ u(0,t) &= h_0(t) && t > 0 \\ u(1,t) &= h_1(t) && t > 0 \\ u(x,0) &= g(x) && 0 < x < 1. \end{aligned}$$

Prove that $u_1 = u_2$ everywhere in the domain $[0, L] \times (0, \infty)$.

[Hint - Let

$$e(t) = \int_0^1 (w(x,t))^2 \, dx$$

for all t, where $w := u_1 - u_2$, and differentiate e with respect to time. Use the example from lectures to guide you.]
