MAT2011 Spring 2009: Unassessed Assignment 1

Please Note: Questions marked with $*$ are intended to be a little bit more challenging.

1.1 Prove that

$$\int_0^\ell u_{xx}(x)u(x)\,\mathrm{d}x \le 0$$

for any twice differentiable function $u(x)$ for which $u(0) = u(\ell) = 0$.

1.2 Find the solution $u(x,t)$ of

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \lambda u, \qquad u(x,0) = a(x).$$

1.3 Find the general solution of

$$u_{xx} + \lambda^2 u = 0$$

where $\lambda \in \mathbb{R}$ is fixed.

1.4 For each of the following PDEs, determine its order and whether it is linear or not. For linear PDEs, state also whether the equation is homogeneous or not; for nonlinear PDEs, circle all term(s) that are not linear.

1. $x^2 u_{xx} + \mathrm{e}^x u = x u_{xyy}$
2. $\mathrm{e}^y u_{xxxx} + \mathrm{e}^x u = -\sin y + 10 x u_y$
3. $y^2 u_{xx} + \mathrm{e}^x u u_x = 2x u_y + u$
4. $y^2 v_{xx} + \mathrm{e}^x = 10 x v_{xy} + v$
5. $u_x u_{xxy} + \mathrm{e}^x u u_y = 5x^2 u_x$
6. $u_t = k^2(u_{xx} + u_{yy}) + f(x,y,t)$

1.5 By direct calculation, check that the function

$$u(x,y) = \sin(x)\frac{\sinh(y)}{\sinh(1)} + \sin(3x)\frac{\sinh(3y)}{\sinh(3)}$$

given in lectures solves the problem

$$\begin{cases} u_{xx} + u_{yy} = 0 & x \in (0,\pi),\ y \in (0,1) \\ u(0,y) = 0 & u(\pi,y) = 0 \\ u(x,0) = 0 & u(x,1) = \sin(x) + \sin(3x). \end{cases}$$

1.6* Suppose $u = u(x,t)$ is smooth and solves the heat equation $u_{xx} = u_t$ in $-\infty < x < \infty$, $t > 0$.

(i) Show that $u_\lambda(x,t) := u(\lambda x, \lambda^2 t)$ also solves the heat equation for each $\lambda \in \mathbb{R}$.

(ii) Show by a direct calculation that

$$v(x,t) := x u_x(x,t) + 2t u_t(x,t)$$

also solves the heat equation.

(iii) Using only the observation that $\frac{d}{d\lambda}u_\lambda$ solves the heat equation (why is this true?), find another proof that v defined above is a solution of the heat equation. [Hint: evaluate $\frac{d}{d\lambda}u_\lambda$ for a particular choice of λ.]
