Assignment 2 MAT2004 Real Analysis II – 2011/12

Web pages:	http://www.maths.surrey.ac.uk/modules/MAT2004.html
Lecturer:	Gianne Derks
Due date:	Wednesday 30 November 2011, start of lecture

- This assignment is for feedback only and is not part of the assessment of the module. You have to hand in solutions as part of your participation requirement.
- Hand in before the start of the lecture on **Wednesday 30 November 2011**.
- There are 6 questions, give full workings with your answers.
- It can be a good idea to discuss possible solution strategies with other students, but you are advised to write the solutions itself independently. However, if you work in a group and also write the solutions together, then it is sufficient to hand in just one piece of work with all names of the group members on it.

Question 1 Determine for each of the following statements whether it is true or false, giving a short proof, an example or a counterexample as appropriate.

(a) If $f : \mathbb{R} \to \mathbb{R}$ is odd and differentiable, then f' is even.

(b) If $f : \mathbb{R} \to \mathbb{R}$ is differentiable and f' is even, then f is odd.

(c) There exists a function $f : \mathbb{R} \to \mathbb{R}$ which is strictly monotonic decreasing and has a point $x_0 \in \mathbb{R}$ with $f'(x_0) = 0$.

Question 2 The numbers $a_1, \ldots, a_n$ are such that $a_1 + 2a_2 + \cdots + na_n = -1$. Show that there is some $c \in (0, 1)$ such that $a_1 + 1 + 4a_2c + \cdots + n^2a_nc^{n-1} = 0$. (Hint: Find a suitable polynomial to apply Rolle's Theorem.)

Question 3 Find an expression for all odd derivatives of $x^2 \sin x$.

Question 4

(a) Find the second order Taylor polynomial for $f(x) = \sqrt[3]{x}$ about $x = x_0$, $x_0 > 0$. For which $x \in \mathbb{R}$ is Taylor's Theorem with remainder valid? Using the second order Taylor polynomial, give an estimate for $\sqrt[3]{1001}$ and the error in the approximation.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be a function for which $f^{(n+1)}$ exists and is continuous and let P_n be the n-th order Taylor polynomial of f about x_0. Show that the function

$$G(x) = \begin{cases} \dfrac{f(x) - P_n(x)}{(x - x_0)^n}, & x \neq x_0; \\ 0, & x = x_0 \end{cases}$$

is differentiable at x_0 and find $G'(x_0)$.

Question 5 Let $f : [a, b] \to \mathbb{R}$ be a bounded function with $f(x) > m > 0$ for $x \in [a, b]$. Let $D = \{x_0, \dots, x_n\}$ be a dissection of $[a, b]$ (hence $x_0 = a$ and $x_n = b$) and define

$$m_i = \inf\{f(x) \mid x_{i-1} < x < x_i\} \quad \text{and} \quad M_i = \sup\{f(x) \mid x_{i-1} < x < x_i\}.$$

Define $g : [a, b] \to \mathbb{R}$ to be $g(x) = \frac{1}{f(x)}$, $x \in [a, b]$. Express the upper and lower sums $\mathcal{U}(g, D)$ and $\mathcal{L}(g, D)$ using m_i, M_i and x_i, $i = 1, \dots, n$.

Question 6 Let $f : [a, b] \to \mathbb{R}$ be a positive continuous function, i.e., $f(x) \geq 0$ for all $x \in [a, b]$.

(a) Show that the function $F : [a, b] \to \mathbb{R}$ defined as $F(x) = \displaystyle\int_a^x f$ for $x \in [a, b]$ is monotone increasing (i.e., $F(x) \geq F(y)$ if $x \geq y$).

(b) Show that if $\displaystyle\int_a^b f = 0$, then $f(x) = 0$ for all $x \in [a, b]$.
