UNIVERSITY OF SURREY©

Faculty of Engineering and Physical Sciences

Department of Mathematics

Undergraduate Programmes in Mathematical Studies

Module MAT2001 — 15 Credits

Numerical and Computational Methods

Level HE2 Examination

Time allowed: $1\frac{1}{2}$ hours

Semester 2, 2010/11

Answer **THREE** questions only
If a candidate attempts more than THREE questions,
then only the best THREE solutions will be taken into account.

Each question carries 20 marks.

Where appropriate the mark carried by an individual
part of a question is indicated in square brackets [ ].

Approved calculators are allowed.

Additional material:
None

Question 1

(a) Use LU-decomposition to solve the following linear system:

$$\begin{bmatrix} 2 & -1 & 3 \\ 4 & 3 & 4 \\ -8 & 19 & -14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ -5 \\ 23 \end{bmatrix}.$$

[13]

(b) State a general formula for $\det(A)$ in terms of L and U and use this formula to compute $\det(A)$ for the matrix in part (a). [3]

(c) The backward difference approximation to the derivative $f'(x_0)$ with error term is given by

$$f'(x_0) = \frac{f(x_0) - f(x_0 - h)}{h} + ch + O(h^2).$$

Prove this result and find the value of the constant c, which is independent of h. [4]

Question 2

(a) Consider the iteration

$$x_{n+1} = g(x_n).$$

Suppose that $x_n \to \alpha$ as $n \to \infty$ for a given initial value x_0.

(i) State the definition of quadratic convergence of this sequence, including the definition of the asymptotic error constant. [3]

(ii) Prove that if $g''(x)$ is continuous, $g'(\alpha) = 0$ and $g''(\alpha) \neq 0$ then the iteration converges quadratically.
What is the value of the asymptotic error constant? [7]

(iii) State Newton's method for finding a solution of the equation $f(x) = 0$.
If g is the iteration function for Newton's method, show that $g'(\alpha) = 0$, where α is a simple root of f (i.e. $f(\alpha) = 0$, $f'(\alpha) \neq 0$).
What is the significance of this result? [6]

(b) Devise an iteration for determining $N^{1/3}$ by applying the Newton iteration to an equation which has this value as a solution. [4]

Question 3

(a) Let $p_{n-1}(x)$ be the polynomial of degree $n-1$ which interpolates the function values $f_0, f_1, \ldots, f_{n-1}$ at the points $x_0, x_1, \ldots, x_{n-1}$ and let $p_n(x)$ be the polynomial of degree n which interpolates these data points together with the additional point (x_n, f_n).

(i) Show that

$$p_n(x) = p_{n-1}(x) + f[x_0, x_1, \ldots, x_n](x - x_0)(x - x_1)\ldots(x - x_{n-1})$$

where $f[x_0, x_1, ..., x_n]$ is a constant.

(ii) Use this result to construct the divided difference form of the interpolating polynomial $p_n(x)$.
(Note: You do not need to derive a formula for the coefficients $f[x_0, x_1, ..., x_n]$.)

(iii) What is the advantage of using the divided difference form of the interpolating polynomial rather than the Lagrange form? [8]

(b) Consider the following data:

$$x_0 = 1, \quad x_1 = 2, \quad x_2 = 4, \qquad f_0 = 9, \quad f_1 = 10, \quad f_2 = 18.$$

For this data, construct (i) the piecewise linear interpolant and (ii) the quadratic interpolating polynomial.
Use both of these approximations to estimate the function at $x = 3$. [12]

Question 4

(a) (i) State Euler's method and the backward Euler method for solving the initial value problem

$$\dot{y} = f(t, y), \qquad y(0) = \alpha.$$

(ii) State one advantage of Euler's method over the backward Euler method.

(iii) Define the local truncation error for Euler's method and show that it is $O(h^2)$, where h is the stepsize. [7]

(b) Apply the backward Euler method to the differential equation

$$\dot{y} = -2ty + \cos t, \qquad y(0) = 2,$$

and rearrange the equation to derive an explicit iteration. [4]

(c) The function $f(x)$ is known at one point x^* in the interval $[a, b]$. Using this value, $f(x)$ can be expressed as

$$f(x) = p_0(x) + f'(\xi(x))(x - x^*)$$

for $x \in [a, b]$, where $p_0(x)$ is the zeroth-order interpolating polynomial $p_0(x) = f(x^*)$ and $\xi(x) \in (a, b)$.

(i) Integrate this expression from a to b to derive a quadrature rule with error term.

(ii) Simplify the error term for the case when $x^* = a$.

(iii) Show that the quadrature rule is exact when $f(x) = c$ for any constant c.

(iv) Find a value of x^* such that the quadrature rule is exact when $f(x) = x$. [9]

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