Summary Notes on Tensors

The following notes contain a summary of the material you need to know on geodesics. Remember as always that these notes are **not a substitute** for reading the book, but simply summarise the key points.

Part C: GEODESICS

In flat space there are two ways of thinking of a straight line. One way of thinking of it is as a curve which does not change direction. The other way of thinking of it is as a curve which gives the shortest distance between two points. In a curved space the first concept generalises to what is called an *affine geodesic* while the second generalises to a *metric geodesic*. We will look at both these concepts and show that they coincide if one uses the metric connection. We will also show how geodesic coordinates may be used to prove tensor identities.

§1 Affine Geodesics

Let

$$x^a = x^a(t), \qquad p \leq t \leq q \tag{1}$$

be the equation of a curve γ. Then

$$V^a(t) = \frac{dx^a}{dt}(t)$$

is a tangent to the curve at the point t.

If $T^{a\ldots}_{\ b\ldots}$ is some tensor field, then we define the *absolute derivative*, $\frac{D}{Dt}$, of the tensor along the curve by

$$\frac{D}{Dt} T^{a\ldots}_{\ b\ldots} = V^c \nabla_c T^{a\ldots}_{\ b\ldots} \tag{2}$$

so that the absolute derivative is the covariant derivative in the direction of the tangent vector V.

If one has a curve whose tangent vector does not change in direction or length as one moves along the curve then

$$\frac{DV^a}{Dt} = 0 \tag{3}$$

However if we still demand that the direction is fixed but allow the length to change then it need only satisfy the weaker condition that any change in the tangent vector is in the same direction as the tangent vector so that

$$\frac{DV^a}{Dt} = \lambda V^a \tag{4}$$

for some scalar field λ. Therefore a curve which does not change direction satisfies the condition

$$V^b \nabla_b V^a = \lambda V^a \tag{5}$$

Now

$$\nabla_b V^a = \frac{\partial V^a}{\partial x^b} + \Gamma^a_{bc} V^c$$

Putting $V^a = \frac{dx^a}{dt}$, this gives

$$\nabla_b V^a = \frac{\partial}{\partial x^b}\left(\frac{dx^a}{dt}\right) + \Gamma^a_{bc}\frac{dx^c}{dt}$$

Hence (5) becomes

$$\frac{dx^b}{dt}\frac{\partial}{\partial x^b}\left(\frac{dx^a}{dt}\right) + \Gamma^a_{bc}\frac{dx^b}{dt}\frac{dx^c}{dt} = \lambda\frac{dx^a}{dt} \tag{6}$$

Since we may use the chain rule to write $\frac{dx^b}{dt}\frac{\partial}{\partial x^b} = \frac{d}{dt}$ then (6) becomes

$$\frac{d^2x^a}{dt^2} + \Gamma^a_{bc}\frac{dx^b}{dt}\frac{dx^c}{dt} = \lambda\frac{dx^a}{dt} \tag{\textbf{6.34}}$$

We call (6.34) the equation of an *affine geodesic* (with a general parameter).

If γ is an affine geodesic it is always possible to find a new parameter s so that $V^a = \frac{dx^a}{ds}$ has constant length, (ie. $g_{ab}V^aV^b = const.$).

With this choice

$$\frac{DV^a}{Ds} = 0$$

so that (6.34) becomes

$$\frac{d^2x^a}{ds^2} + \Gamma^a_{bc}\frac{dx^b}{ds}\frac{dx^c}{ds} = 0 \tag{\textbf{6.37}}$$

We call such a parameter an *affine parameter* and equation (6.37) that of an affinely parameterised affine geodesic.

§2 Metric Geodesics

The Calculus of Variations

We first look at the problem of finding the maximum (or minimum) of an integral which depends upon an arbitrary function $x(t)$ and its derivative $\dot{x}(t)$. We therefore consider integrals of the form

$$I = \int_{t=p}^{q} L(x(t), \dot{x}(t))dt \tag{7}$$

where $L(x, \dot{x})$ is a given function of two variables and $x(t)$ is a function we are free to choose subject to the boundary conditions that

$$x(p) = k_1 \quad \text{and } x(q) = k_2 \quad \text{for fixed constants } k_1 \text{ and } k_2. \tag{8}$$

Let $x(t)$ be some function which satisfies these boundary conditions and $\eta(t)$ be an arbitrary function which satisfies the boundary conditions $\eta(p) = 0$, $\eta(q) = 0$. Then for any value of ϵ the function

$$\tilde{x}(t) = x(t) + \epsilon\eta(t)$$

also satisfies the required boundary conditions (8). Note also that when $\epsilon = 0$ then $\tilde{x}(t) = x(t)$.

Suppose now that (somehow) we have found the function $x(t)$ which gives the minimum value of the integral. Then if we define

$$f(\epsilon) = \int_{t=p}^{q} L(x(t) + \epsilon\eta(t), \dot{x}(t) + \epsilon\dot{\eta}(t))dt \tag{9}$$

We see that this must have a minimum at $\epsilon = 0$, since $x(t)$ gives the minimum value of the integral. Thus

$$f'(0) = 0 \tag{10}$$

However

$$\begin{aligned}
\frac{df}{d\epsilon} &= \int_{t=p}^{q} \frac{dL}{d\epsilon} dt \\
&= \int_{t=p}^{q} \left(\frac{\partial L}{\partial x}\eta(t) + \frac{\partial L}{\partial \dot{x}}\dot{\eta}(t) \right) dt
\end{aligned}$$

Integrating the second term by parts we get

$$\begin{aligned}\frac{df}{d\epsilon} &= \left[\frac{\partial L}{\partial \dot{x}}\eta(t)\right]_{t=p}^{q} + \int_{t=p}^{q}\left(\frac{\partial L}{\partial x}\eta(t) - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\eta(t)\right)dt \\ &= \int_{t=p}^{q}\left(\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right)\eta(t)dt\end{aligned}$$

Since the first term vanishes because $\eta(p) = 0$ and $\eta(q) = 0$.

Therefore $f'(0) = 0$ implies that

$$\int_{t=p}^{q}\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x}\right)\eta(t)dt = 0 \tag{11}$$

for *all possible* choices of η. The only way that this can be true is if

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \tag{12}$$

Therefore if $x(t)$ is to give the minimum value of the integral I then it must satisfy the differential equation (12). We call this equation the Euler-Lagrange equation for I.

We can also consider the more general case where the function L depends upon the curve $x^a(t)$ and its derivative $\dot{x}^a(t)$ so that

$$I = \int_{t=p}^{q} L(x^a(t), \dot{x}^a(t))dt \tag{13}$$

Then in this case the maxima (and minima) of I occur when $x^a(t)$ satisfies the Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^a} - \frac{\partial L}{\partial x^a} = 0 \qquad a = 1, \ldots, n \tag{14}$$

The equation of a metric geodesic

In order to calculate the equation of a metric geodesic we wish to minimise

$$I = \int ds \tag{15}$$

We therefore take

$$L(x^a(t), \dot{x}^a(t)) = \left(g_{bc}\dot{x}^b(t)\dot{x}^c(t)\right)^{1/2} = \frac{ds}{dt} \tag{16}$$

(See equation (25) of part A).

With this choice of L we have

$$\begin{aligned}\frac{\partial L}{\partial \dot{x}^a} &= (g_{ab}\dot{x}^b)(g_{cd}\dot{x}^c\dot{x}^d)^{-1/2}\\ &= (g_{ab}\dot{x}^b)/\frac{ds}{dt}\end{aligned}$$

Differentiating with respect to t gives:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^a}\right) = (g_{ab,c}\dot{x}^c\dot{x}^b)/\frac{ds}{dt} + (g_{ab}\ddot{x}^b)/\frac{ds}{dt} - (g_{ab}\dot{x}^b)\frac{d^2s}{dt^2}/(\frac{ds}{dt})^2$$

On the other hand

$$\begin{aligned}\frac{\partial L}{\partial x^a} &= \frac{1}{2}g_{bc,a}\dot{x}^b\dot{x}^c(g_{de}\dot{x}^d\dot{x}^e)^{-1/2}\\ &= \frac{1}{2}g_{bc,a}\dot{x}^b\dot{x}^c/\frac{ds}{dt}\end{aligned}$$

In order to simplify these expression we now choose to parameterise the curve by the length s. If we do this then $t = s$ and hence

$$\frac{ds}{dt} = 1, \qquad \frac{d^2s}{dt^2} = 0 \tag{17}$$

So that

$$\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{x}^a}\right) = (g_{ab,c}\dot{x}^c\dot{x}^b) + g_{ab}\ddot{x}^b$$

and

$$\frac{\partial L}{\partial x^a} = \frac{1}{2}g_{bc,a}\dot{x}^b\dot{x}^c$$

The Euler-Lagrange equations are then given by

$$g_{ab}\ddot{x}^b + (g_{ab,c} - \frac{1}{2}g_{bc,a})\dot{x}^b\dot{x}^c = 0 \tag{18}$$

Multiplying by g^{ad} this gives

$$g_{ab}g^{ad}\ddot{x}^b + \frac{1}{2}g^{ad}(g_{ab,c} + g_{ac,b} - g_{bc,a})\dot{x}^b\dot{x}^c = 0 \tag{19}$$

So if we let Γ^a_{bc} be the metric connection (see part B equation (6.71)) we may write this as

$$\ddot{x}^d + \Gamma^d_{bc}\dot{x}^b\dot{x}^c = 0 \tag{20}$$

which is the equation of an affinely parameterised, affine geodesic. Hence we see that if we use the metric connection, an affine geodesic is a metric geodesic and length is an affine parameter.

We end this section by showing that we get the *same* Euler-Lagrange equations by again choosing the length s as our affine parameter, but now minimising

$$I = \int g_{bc}\dot{x}^b\dot{x}^c ds \tag{21}$$

(Note the absence of the square root).

In this case we have

$$L = g_{bc}\dot{x}^b\dot{x}^c$$

Hence

$$\frac{\partial L}{\partial \dot{x}^a} = 2g_{ab}\dot{x}^b$$

so that

$$\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{x}^a}\right) = 2g_{ab,c}\dot{x}^c\dot{x}^b + 2g_{ab}\ddot{x}^b$$

Also

$$\frac{\partial L}{\partial x^a} = g_{bc,a}\dot{x}^b\dot{x}^c$$

So the Euler-Lagrange equations are

$$2g_{ab}\ddot{x}^b + 2g_{ab,c}\dot{x}^c\dot{x}^b - g_{bc,a}\dot{x}^b\dot{x}^c = 0 \tag{22}$$

which apart from the irrelevant overall factor of 2 agrees with (18). It is this form of L (without the square root) that we use in practise to calculate geodesics.

Example

In plane polar coordinates the metric of 2-dimensional Euclidean space is given by

$$ds^2 = dr^2 + r^2 d\phi^2 \tag{23}$$

So that

$$L = \dot{r}^2 + r^2\dot{\phi}^2 \tag{24}$$

We get two Euler-Lagrange equations, one for the r coordinate and one for the ϕ coordinate. We first consider the r-equation.

$$\begin{aligned} \frac{\partial L}{\partial \dot{r}} &= 2\dot{r} \\ \frac{d}{ds}\left(\frac{\partial L}{\partial \dot{r}}\right) &= 2\ddot{r} \\ \frac{\partial L}{\partial r} &= 2r\dot{\phi}^2 \end{aligned}$$

So that the Euler-Lagrange equation for the r coordinate is

$$\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \quad \Rightarrow \quad \ddot{r} - r\dot{\phi}^2 = 0 \tag{25}$$

We now consider the ϕ-equation

$$\begin{aligned} \frac{\partial L}{\partial \dot{\phi}} &= 2r^2\dot{\phi} \\ \frac{d}{ds}\left(\frac{\partial L}{\partial \dot{\phi}}\right) &= 4r\dot{r}\dot{\phi} + 2r^2\ddot{\phi} \\ \frac{\partial L}{\partial \phi} &= 0 \end{aligned}$$

So that the Euler-Lagrange equation for the ϕ coordinate is

$$\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0 \quad \Rightarrow \quad \ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} = 0 \tag{26}$$

On the other hand if we choose $(x^1, x^2) = (r, \phi)$, a (rather long) direct calculation using formula (6.71) from part B shows

$$\Gamma^1_{22} = -r, \quad \Gamma^2_{21} = \Gamma^2_{12} = \frac{1}{r}, \quad \Gamma^a_{bc} = 0, \text{ otherwise}$$

So that (25) and (26) are just

$$\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0 \tag{27}$$

as claimed.

§3 Geodesic Coordinates

In proving tensor identities it is often useful to work in a *special* coordinate system which simplifies the calculation and then use the fact that one is working with tensors to deduce that the identity is true in *any* coordinate system. A particularly useful choice of coordinates is that given by a *geodesic coordinate* system. Given any point P on a manifold it is possible to introduce local coordinates so that

$$g_{ab,c} \overset{*}{=} 0 \quad \text{at } P \tag{28}$$

where the symbol $\overset{*}{=}$ indicates that the result is only true in geodesic coordinates and might not be true in an other coordinate system.

It is important to realise that since (28) is only true at the point P we **cannot** differentiate it, so that in general $g_{ab,cd}$ will not be zero at P even if one works in geodesic coordinates.

A consequence of (28) is that

$$\Gamma^a_{bc} \overset{*}{=} 0 \quad \text{at } P \tag{29}$$

so that near P the coordinate lines are approximately geodesics.

It also follows from (28) that

$$g^{ab}{}_{,c} \overset{*}{=} 0 \quad \text{at } P \tag{30}$$

We can use (29) and (30) to obtain some simple expressions for the curvature in geodesic coordinates at P. We first note that substituting for Γ^a_{bc} in (6.39) and using (29) gives

$$R^a{}_{bcd} \overset{*}{=} \Gamma^a_{bd,c} - \Gamma^a_{bc,d} \quad \text{at } P \tag{31}$$

Then differentiating (6.71) and using (30) gives

$$\Gamma^a_{bc,d} \overset{*}{=} \frac{1}{2} g^{ae}(g_{ec,bd} + g_{eb,cd} - g_{bc,ed}) \quad \text{at } P \tag{32}$$

Substituting in (31) now gives

$$R^a{}_{bcd} \stackrel{*}{=} \frac{1}{2} g^{ae}(g_{ed,bc} + g_{bc,ed} - g_{bd,ec} - g_{ec,bd}) \quad \text{at } P \tag{33}$$

Lowering the index a gives

$$R_{abcd} \stackrel{*}{=} \frac{1}{2}(g_{ad,bc} + g_{bc,ad} - g_{bd,ac} - g_{ac,bd}) \quad \text{at } P \tag{34}$$

This equation may be used to establish a number of identities involving $R^a{}_{bcd}$. For example from (34) it easily follows that

$$R_{abcd} - R_{cdab} \stackrel{*}{=} 0 \quad \text{at } P \tag{35}$$

However this implies that

$$R_{abcd} - R_{cdab} = 0 \quad \text{at } P \tag{36}$$

in *any* coordinate system (since (35) is a *tensor* equation). Finally since the point P was arbitrary we can conclude that

$$R_{abcd} - R_{cdab} = 0 \quad \text{everywhere} \tag{37}$$

which establishes equation (6.79) of part B. In a similar way one can use geodesic coordinates and (34) to prove that

$$R_{abcd} + R_{acdb} + R_{adbc} = 0 \tag{38}$$

(See Exercise 6.23)