Summary Notes on Tensors

The following notes contain a summary of the material you need to know on tensor calculus. Remember however that these notes are **not a substitute** for reading the book, but simply summarise the key points.

Part B: TENSOR CALCULUS

§1 The covariant derivative

Recall that if ϕ is a scalar field then the derivative $N_a = \phi_a$ is a covariant vector. We now ask the following question:

If X_a is a covariant vector is $S_{ab} = X_{a,b}$ a type $(0,2)$ tensor?

To answer this question we calculate S in the x' coordinates. By definition

$$\begin{aligned}
S'_{ab} &= \frac{\partial}{\partial x'^b}\left(X'_a\right) \\
&= \frac{\partial}{\partial x'^b}\left(\frac{\partial x^c}{\partial x'^a}X_c\right) \\
&= \frac{\partial^2 x^c}{\partial x'^b \partial x'^a}X_c + \frac{\partial x^c}{\partial x'^a}\frac{\partial}{\partial x'^b}X_c \\
&= \frac{\partial^2 x^c}{\partial x'^b \partial x'^a}X_c + \frac{\partial x^c}{\partial x'^a}\frac{\partial x^d}{\partial x'^b}\frac{\partial}{\partial x^d}X_c \\
&= \frac{\partial^2 x^c}{\partial x'^b \partial x'^a}X_c + \frac{\partial x^c}{\partial x'^a}\frac{\partial x^d}{\partial x'^b}X_{c,d} \\
&= \frac{\partial x^c}{\partial x'^a}\frac{\partial x^d}{\partial x'^b}S_{cd} + \frac{\partial^2 x^c}{\partial x'^b \partial x'^a}X_c
\end{aligned}$$

However for a *tensor* field T_{ab} one has

$$T'_{ab} = \frac{\partial x^c}{\partial x'^a}\frac{\partial x^d}{\partial x'^b}T_{cd}$$

So that $X_{a,b}$ fails to transform as a tensor due to the term involving the second derivative of the transformation.

In order to obtain a tensor when one differentiates, one needs some extra structure which cancels out this extra term when one changes to a new coordinate system.

We therefore define the *covariant derivative* (which we denote with a semicolon rather than a comma) by

$$X_{a;b} = X_{a,b} - \Gamma^{c}_{ab} X_c \tag{1}$$

Where Γ^a_{bc} is an object called the connection. We will see a way of specifying the connection in terms of the metric in the next section, but for the moment the important thing to note is that Γ^a_{bc} transforms according to

$$\Gamma'^a_{bc} = \frac{\partial x'^a}{\partial x^d} \frac{\partial x^e}{\partial x'^b} \frac{\partial x^f}{\partial x'^c} \Gamma^d_{ef} + \frac{\partial x'^a}{\partial x^d} \frac{\partial^2 x^d}{\partial x'^b \partial x'^c} \tag{\textbf{6.24}}$$

Therefore Γ^a_{bc} also has a second derivative term which prevents it transforming as a tensor. However if one considers the combination in equation (1) one finds that the 'bad' terms exactly cancel out and the covariant derivative does indeed transform as a tensor so that

$$X'_{a;b} = \frac{\partial x^c}{\partial x'^a} \frac{\partial x^d}{\partial x'^b} X_{c;d}$$

In the same way one can define the covariant derivative of a (contravariant) vector

$$Y^a_{\ ;b} = Y^a_{\ ,b} + \Gamma^a_{bc} Y^c$$

Again one can show that this combination transforms as a tensor, but this time one needs the alternative form of the transformation law of Γ^a_{bc} given by

$$\Gamma'^a_{bc} = \frac{\partial x'^a}{\partial x^d} \frac{\partial x^e}{\partial x'^b} \frac{\partial x^f}{\partial x'^c} \Gamma^d_{ef} + \frac{\partial x^d}{\partial x'^b} \frac{\partial x^e}{\partial x'^c} \frac{\partial^2 x'^a}{\partial x^d \partial x^e} \tag{\textbf{6.23}}$$

These two expression may be shown to be equivalent by differentiating

$$\frac{\partial x'^a}{\partial x^d} \frac{\partial x^d}{\partial x'^c} = \delta^a_c$$

with respect to x'^b (see exercise 6.3).

It is also possible to take the covariant derivative of a general tensor. One simply requires a Γ term for each index.

Example

$$T^a_{bc;d} = T^a_{bc,d} + \Gamma^a_{ed} T^e_{bc} - \Gamma^e_{bd} T^a_{ec} - \Gamma^e_{cd} T^a_{be}$$

(see also equation (6.27) of the set book).

Notation

Rather than write a partial derivative using a comma like ${X^a}_{,b}$ we will sometimes use the partial derivative symbol and write $\partial_b X^a$

In the same way rather than write a covariant derivative using a semicolon like ${X^a}_{;b}$ we will sometimes use the inverted triangle symbol and write $\nabla_b X^a$.

It is also useful to have a symbol for $V^b \nabla_b X^a$ which represents the covariant derivative of X^a in the direction of the vector V. We will sometimes write this as $\nabla_V X^a$

Torsion

As we have seen the connection Γ^a_{bc} does not transform as a tensor however if one looks at

$$T^a_{bc} = \Gamma^a_{bc} - \Gamma^a_{cb} \tag{2}$$

then the second derivative terms cancel and one finds that T^a_{bc} transforms as a tensor. We call the tensor T^a_{bc} the *torsion* of the connection. Mathematically it is possible to consider connections with non-zero torsion (and in some physical theories it is used to describe the spin of particles). However the attitude we will take is to work with a *torsion free* connection, and if we wish to describe any physics which depends on torsion explicitly add it in the form of an additional tensor field T^a_{bc}. For the rest of the unit you may therefore assume that

$$\Gamma^a_{bc} = \Gamma^a_{cb} \tag{\textbf{6.28}}$$

Note that since the torsion is a tensor, if it vanishes in one coordinate system then it vanishes in every coordinate system so that (6.28) is true in all coordinate systems even though Γ^a_{bc} is not a tensor.

The metric connection

Given a metric g_{ab} it is possible to obtain a unique formula for Γ^a_{bc} in terms of the metric by demanding that it also satisfies the condition

$$\nabla_c g_{ab} = 0 \tag{\textbf{6.73}}$$

This is very natural as it means we treat the covariant derivative of both the contravariant form X^a, and covariant form X_a of a vector on an equal footing.

We can either differentiate X^a and then lower the index with g_{ab} or lower the index first and then differentiate. In other words

$$g_{ab} \nabla_c X^a = \nabla_c X_a \tag{3}$$

We will also see later that (6.73) means that the two possible generalisations of a straight line to curved space – firstly as a curve which doesn't change direction (an affine geodesic) and secondly as the shortest distance between two points (a metric geodesic) – coincide as the single concept of geodesic.

A connection which satisfies both (6.28) and (6.73) is called the Levi-Cevita connection or *metric connection*. Using (6.28) and (6.73) one can obtain the following formula for the metric connection

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad} (\partial_b g_{dc} + \partial_c g_{db} - \partial_d g_{bc}) \tag{\textbf{6.71}}$$

Note in some books a special symbol is used for the components of the metric connection. These are called the Christoffel symbols of the first and second kind and are given in equations (6.62) and (6.64) of the set book.

§2 The Riemann curvature tensor

If one uses the partial derivative ∂_a then the order of differentiation does not matter. So for example

$$\partial_c \partial_d X^a - \partial_d \partial_c X^a = 0 \tag{4}$$

However, if one works with the covariant derivative then (in general) this is not true so that

$$\nabla_c \nabla_d X^a - \nabla_d \nabla_c X^a \neq 0 \tag{5}$$

Although the right hand side does not vanish it only depends linearly upon the vector X (since the terms involving the second derivative of X vanish due to (4) and those involving the first derivative vanish due to the torsion free condition (6.28)). The right hand side of (5) therefore has the form $L^a_b X^b$. However the linear transformation L depends upon the values of c and d so that we may write

$$\nabla_c \nabla_d X^a - \nabla_d \nabla_c X^a = R^a{}_{bcd} X^b \tag{\textbf{6.38}}$$

We finally remark that the left hand side is a tensor (since the covariant derivative of a tensor is a tensor), so the right hand side is also a tensor.

Since X is an arbitrary vector this means that $R^a{}_{bcd}$ is also a tensor. Thus (6.38) defines a type $(1,3)$ tensor called the *Riemann curvature tensor*.

If we use the formula (1) to write the covariant derivative in terms of the partial derivative and the connection Γ^a_{bc} one can obtain an expression for $R^a{}_{bcd}$ in terms of the connection and its derivatives.

$$R^a{}_{bcd} = \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^e_{bd}\Gamma^a_{ec} - \Gamma^e_{bc}\Gamma^a_{ed} \qquad (\mathbf{6.39})$$

Note that it is not at all obvious from the above expression that $R^a{}_{bcd}$ are the components of a tensor even though we know from (6.38) that it must be. Since equation (6.71) gives an expression for Γ^a_{bc} in terms of the metric g_{ab} and its partial derivatives, one can in principle write down an expression for the curvature tensor in terms of g_{ab} and its first and second partial derivatives. However the resulting expression is rather long and not very helpful so we will not give it here.

Symmetries of the Curvature tensor

Since the curvature tensor $R^a{}_{bcd}$ has four different indices, each of which can take n different values, the curvature tensor has 4^n different components. So that in 4-dimensions it has $4^4 = 256$ different components. However the curvature tensor also satisfies a number of algebraic identities so that not all these components are independent.

For example it is easy to see from (6.38) that it is antisymmetric on the last two indices and thus

$$R^a{}_{bcd} = -R^a{}_{bdc} \qquad (\mathbf{6.77})$$

It is not so obvious, (but is not too hard to prove), that the curvature also satisfies the identity

$$R^a{}_{bcd} + R^a{}_{cdb} + R^a{}_{dbc} = 0 \qquad (\mathbf{6.78})$$

Finally if one lowers the first index to obtain $R_{abcd} = g_{ae}R^e{}_{bcd}$ then the tensor is symmetric on interchanging the first and last pair of indices so that

$$R_{abcd} = R_{cdab} \qquad (\mathbf{6.79})$$

Note combining (6.79) and (6.77) gives

$$R_{abcd} = -R_{bacd}$$

So the completely covariant form of the curvature tensor is also antisymmetric on the first two indices as well as the last two.

All these symmetries reduce the number of independent components of the curvature tensor to $\frac{1}{12}n^2(n^2-1)$ which in 4-dimensions is 20 (rather than the 256 we started with).

As well as these algebraic symmetries the curvature tensor satisfies an important differential identity called the *Bianchi identity*. This says that

$$\nabla_a R^d{}_{ebc} + \nabla_c R^d{}_{eab} + \nabla_b R^d{}_{eca} = 0 \qquad (\mathbf{6.82})$$

The Ricci curvature, Scalar Curvature and Einstein tensor

If we start with the curvature tensor R_{abcd} it is possible to construct new tensors by contracting on two of the indices. For example we could look at $g^{ab}R_{abcd}$. However since the first term is symmetric on a and b, while the second term is antisymmetric on a and b this contraction gives zero (see exercise 5.11). So instead we consider the contraction on the first and third index

$$R_{bd} = g^{ac}R_{abcd} \qquad (\mathbf{6.83})$$

In general this does not vanish but gives a tensor R_{ab} called the *Ricci tensor* or Ricci curvature. The identity (6.79) implies that this tensor is symmetric, so that

$$R_{ab} = R_{ba}$$

Because of this symmetry this tensor has 10 (rather than 16) independent components in 4-dimensions. One can also consider contraction on any other pair of indices, but these either vanish (eg on 3 and 4) or else give the same answer as the Ricci tensor (eg on 2 and 4). Physically the Ricci tensor is important because in Einstein's theory of General Relativity it is the part of the curvature which couples to the matter.

It is also possible to contract the two indices of the Ricci tensor to obtain a scalar

$$R = g^{ab}R_{ab} \qquad (\mathbf{6.84})$$

which is called the *scalar curvature*.

One can then combine the Ricci curvature with the scalar curvature and the metric to form the *Einstein tensor* G_{ab} which is defined by

$$G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R \qquad (\mathbf{6.85})$$

The reason we add on the second term to the Ricci curvature is to obtain a quantity with vanishing divergence

$$\nabla_a G^a{}_b = 0 \tag{6.86}$$

This identity is called the *contracted Bianchi identity* and will turn out to be important physically, because when combined with Einstein's equations it implies the conservation of energy and momentum.

The Weyl Curvature

In 4-dimensions we have seen that the curvature has 20 independent components, 10 of these components can be described using the Ricci curvature which is obtained by contracting (or taking the trace) on a pair of indices. The remaining 10 components may be described in terms of a tensor C_{abcd} which has the same symmetries as the Riemann curvature tensor but also has the property that it is completely trace free (this means that if one contracts on *any* pair of indices one gets zero). In 4-dimensions the Weyl curvature may be defined in terms of the Riemann, Ricci and scalar curvature by the equation

$$C_{abcd} = R_{abcd} + \frac{1}{2}(g_{ad}R_{cb} + g_{bc}R_{da} - g_{ac}R_{db} - g_{bd}R_{ca}) + \frac{1}{6}(g_{ac}g_{db} - g_{ad}g_{bc})R \tag{6.87}$$

Physically the Weyl curvature is important because it is the only part of the curvature which is non-zero in the absence of matter. It is therefore the part of the curvature which describes gravitational radiation.

Mathematically the Weyl tensor is also important because it is conformally invariant. That is, if one defines a new metric

$$\bar{g}_{ab} = \Omega^2 g_{ab} \tag{6.90}$$

where Ω is a scalar field, then the Weyl curvature of the new metric is the same as the Weyl curvature of the old metric (when written in the $(1,3)$ form). That is:

$$\bar{C}^a{}_{bcd} = C^a{}_{bcd} \tag{6.91}$$