A brief introduction to tensor algebra

MATH3006
Relativity, black holes and cosmology

Vectors

Tensors generalise the concept of vectors.

Let $\boldsymbol{v}$ be a vector and $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ the standard Cartesian basis. Then

$$\boldsymbol{v} = v_1 \boldsymbol{i} + v_2 \boldsymbol{j} + v_3 \boldsymbol{k}$$

and we call v_1, v_2 and v_3 the components of the vector in this basis.

We can make this more compact by introducing

$$\hat{\boldsymbol{e}}_1 = \boldsymbol{i} \; , \; \hat{\boldsymbol{e}}_2 = \boldsymbol{j} \; , \; \hat{\boldsymbol{e}}_3 = \boldsymbol{k}$$

$$\hat{\boldsymbol{e}}_1 \cdot \hat{\boldsymbol{e}}_1 = 1 \; , \; \hat{\boldsymbol{e}}_1 \cdot \hat{\boldsymbol{e}}_2 = 0 \; , \text{ etcetera}$$

Using the Einstein summation convention we then have

$$\boldsymbol{v} = \sum_{i=1}^{3} v_i \hat{\boldsymbol{e}}_i = v_i \hat{\boldsymbol{e}}_i$$

The natural basis

The Cartesian basis is orthonormal.

It is often more natural to work with a different basis.

Consider a system described by coordinates x^1, x^2, and x^3. If $\boldsymbol{r} = \boldsymbol{r}(x^1, x^2, x^3)$ is the vector connecting the origin to the point with coordinates (x^1, x^2, x^3) , then we define the natural basis as

$$\boldsymbol{e}_1 = \frac{\partial \boldsymbol{r}}{\partial x^1} \;,\; \boldsymbol{e}_2 = \frac{\partial \boldsymbol{r}}{\partial x^2} \;,\; \boldsymbol{e}_3 = \frac{\partial \boldsymbol{r}}{\partial x^3} \;\Leftrightarrow\; \boldsymbol{e}_i = \frac{\partial \boldsymbol{r}}{\partial x^i}$$

and we call v^1, v^2 and v^3 the components of the vector in this basis. In this case we have

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_j = 0 \;,\; i \neq j$$

but typically

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_i \neq 1$$

In the general (orthogonal) case we have

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_i = h_i^2$$

where h_i is called the scale factor.

Example.

For spherical polar coordinates we have

$$\boldsymbol{e}_r \cdot \boldsymbol{e}_r = 1\,,\ \boldsymbol{e}_\theta \cdot \boldsymbol{e}_\theta = r^2\,,\ \boldsymbol{e}_\varphi \cdot \boldsymbol{e}_\varphi = r^2 \sin^2\theta$$

In the natural basis we have

$$\boldsymbol{v} = v^1 \boldsymbol{e}_1 + v^2 \boldsymbol{e}_2 + v^3 \boldsymbol{e}_3 = v^i \boldsymbol{e}_i$$

This kind of basis – known as a coordinate basis – is used in most tensor calculations.

It now makes a difference if indices are up or down.

We need to distinguish between vectors and covectors.

Covectors

Covectors are dual to vectors. If you “apply” a covector to a vector you get a number. That is, if α is a covector and $\boldsymbol{v}$ is a vector, then

$$\alpha(\boldsymbol{v}) = \text{ number}$$

Given a natural basis $\boldsymbol{e}_i$ we obtain the dual basis by requiring

$$\boldsymbol{e}^i(\boldsymbol{e}_j) = \delta^i_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Let the covector have components α_i in the dual basis. Then it follows that

$$\begin{aligned} \alpha(\boldsymbol{v}) = \alpha_i \boldsymbol{e}^i(v^j \boldsymbol{e}_j) = \alpha_i v^j \boldsymbol{e}^i(\boldsymbol{e}_j) &\qquad \text{by linearity} \\ = \alpha_i v^j \delta^i_j &\qquad \text{definition of dual basis} \\ = \alpha_i v^i &\qquad \text{definition of Kronecker delta} \end{aligned}$$

Contract a vector and a covector to get a number.

The metric

The metric is a (symmetric) object g_{ij} such that, if $\boldsymbol{v}$ and $\boldsymbol{w}$ are vectors, then

$$g_{ij} v^i w^j$$

is a number which does not depend on the coordinate system.

In (flat space) Cartesian coordinates we have

$$g_{ij} \doteq \delta_{ij} \quad (\doteq \text{ means that it is only true in this coordinate system})$$

In general the metric is such that

$$g_{ij} = \frac{\partial \boldsymbol{r}}{\partial x^i} \cdot \frac{\partial \boldsymbol{r}}{\partial x^j} = \boldsymbol{e}_i \cdot \boldsymbol{e}_j$$

We can use the metric to raise and lower indices. If $\boldsymbol{v}=v^i\boldsymbol{e}_i$ then we define the corresponding covector $v_i\boldsymbol{e}^i$ using the dual basis and

$$v_i = g_{ij} v^j$$

The inverse

In general, the inverse of the metric follows from

$$g^{ij} g_{jk} = g^i_k \equiv \delta^i_k$$

For an orthogonal coordinate system the metric is diagonal, i.e. we have

$$\begin{cases} g_{ij} = 0 & i \neq j \\ g_{ii} = h_i^2 & i = j \end{cases} \quad \Longleftrightarrow \quad g_{ij} = \begin{pmatrix} h_1^2 & 0 & 0 \\ 0 & h_2^2 & 0 \\ 0 & 0 & h_3^2 \end{pmatrix}$$

Then it is easy to see that the inverse is

$$g^{ij} = \begin{pmatrix} 1/h_1^2 & 0 & 0 \\ 0 & 1/h_2^2 & 0 \\ 0 & 0 & 1/h_3^2 \end{pmatrix}$$

We also see that $v_i = g_{ij} v^j = h_i^2 v^i$ (note: no summation here)

A useful picture?

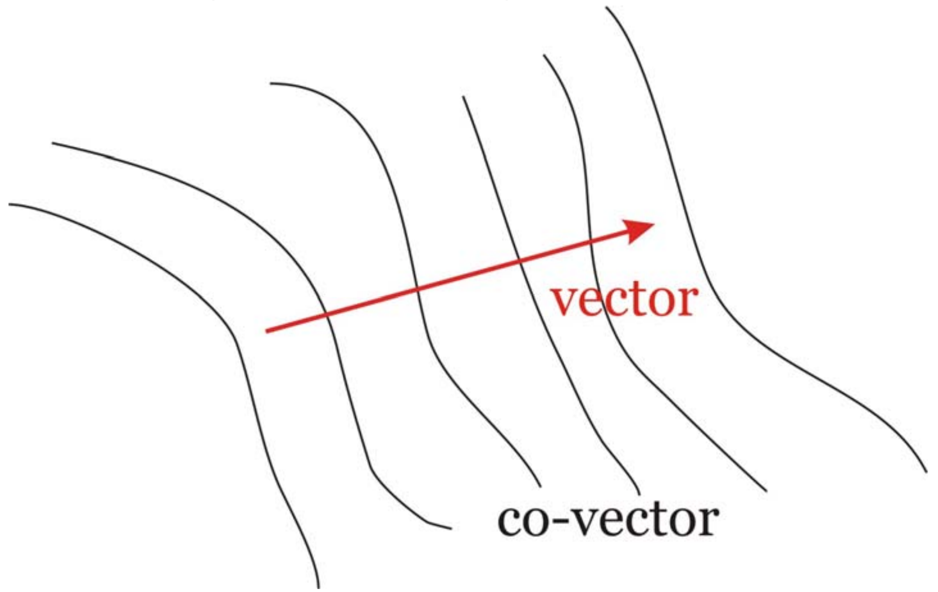
We are all familiar with the picture of a vector as an arrow pointing in a certain direction. What is the corresponding image of a co-vector?

First of all, it can’t be an arrow like a vector. According to our definition, we need a co-vector acting on a vector to produce a number.

Two vectors can’t do this. In order to produce a number from two vectors you need a metric.

A useful picture?

A good example of a co-vector is the gradient ∇_i. The gradient is closely related to the spacing of level surfaces of a scalar field. This suggests that we could think of a co-vector as a set of surfaces, with its magnitude increasing if the spacing between the surfaces is smaller.



The "number" produced by a co-vector acting on a vector represents the "number of surfaces that are pierced by" the vector.

Cartesian tensors

As an example of index gymnastics (and an excuse to discuss some additional concepts), let us use this new formalism to derive some well-know identities from vector calculus.

First of all, it is more or less obvious that the metric allows us to work out scalar products;

$$\boldsymbol{v}\cdot\boldsymbol{v} \quad \Leftrightarrow \quad v^i v_i = g_{ij} v^i v^j$$

To define the cross product, we need to introduce the totally anti-symmetric object

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if } [i,j,k]=[1,2,3] \text{ or cyclic permutation} \\ -1 & \text{if permuation is not cyclic} \\ 0 & \text{if any two indices are the same} \end{cases}$$

Then one can show that

$$\boldsymbol{v}\times\boldsymbol{w} \quad \Leftrightarrow \quad \varepsilon^{ijk} v_j w_k$$

Let us now focus on Cartesian tensors, in which case the basis is constant. Then we introduce the usual gradient as

$$\nabla_i \doteq \frac{\partial}{\partial x^i} = \partial_i$$

It follows that

$$\nabla \cdot \boldsymbol{v} \quad \Leftrightarrow \quad \nabla_i v^i \qquad \text{and} \qquad \nabla \times \boldsymbol{v} \quad \Leftrightarrow \quad \varepsilon^{ijk} \nabla_j v_k$$

Now it is trivial to show that

$$\nabla \cdot (\nabla \times \boldsymbol{v}) \;\Rightarrow\; \nabla_i (\varepsilon^{ijk} \nabla_j v_k) = \varepsilon^{ijk} \nabla_i \nabla_j v_k = 0$$

since the Levi-Civita tensor is anti-symmetric.

Explicitly;

$$\varepsilon^{ijk} \nabla_i \nabla_j = \frac{1}{2}\left(\varepsilon^{ijk} \nabla_i \nabla_j + \varepsilon^{jik} \nabla_j \nabla_i\right) = \frac{1}{2}\left(\varepsilon^{ijk} + \varepsilon^{jik}\right) \nabla_i \nabla_j = 0$$

Noting the identity

$$\varepsilon^{ijk}\varepsilon_{klm} = \delta^i_l\delta^j_m - \delta^i_m\delta^j_l$$

we can readily derive more complicated vector identities, e.g.

$$\varepsilon^{ijk}\nabla_j(\varepsilon_{klm}\nabla^l v^m) = \varepsilon^{ijk}\varepsilon_{klm}\nabla_j\nabla^l v^m =$$

$$(\delta^i_l\delta^j_m - \delta^i_m\delta^j_l)\nabla_j\nabla^l v^m = \nabla_j\nabla^i v^j - \nabla_j\nabla^j v^m = \nabla^i(\nabla_j v^j) - \nabla^2 v^i$$

$$\boxed{\nabla\times(\nabla\times\boldsymbol{v}) = \nabla(\nabla\cdot\boldsymbol{v}) - \nabla^2\boldsymbol{v}}$$

At this point it is quite easy to write various equations from physics on tensorial form. This has the advantage that they become coordinate independent.

Fluid dynamics :

$$\begin{aligned} \partial_t \rho + \nabla \cdot (\rho \boldsymbol{v}) &= 0 \\ \partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} + \frac{1}{\rho} \nabla p &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \partial_t \rho + \nabla_i (\rho v^i) &= 0 \\ \partial_t v^i + (v^j \nabla_j) v^i + \frac{1}{\rho} \nabla^i p &= 0 \end{aligned}$$

Maxwell's equations:

$$\begin{aligned} \nabla \cdot \boldsymbol{D} &= 4\pi\sigma \\ \nabla \cdot \boldsymbol{B} &= 0 \\ \partial_t \boldsymbol{D} - \nabla \times \boldsymbol{H} &= -4\pi \boldsymbol{j} \\ \partial_t \boldsymbol{B} + \nabla \times \boldsymbol{E} &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \nabla_i D^i &= 4\pi\sigma \\ \nabla_i B^i &= 0 \\ \partial_t D^i - \varepsilon^{ijk} \nabla_j H_k &= -4\pi j^i \\ \partial_t B^i + \varepsilon^{ijk} \nabla_j E_k &= 0 \end{aligned}$$

Towards four dimensions...

In relativity, time is considered on an equal footing with the spatial coordinates leading to the key concept of spacetime.

The good news is that all the tensor concepts we have discussed extend immediately to four dimensions. We only need to replace

$i, j, k, \ldots$ ranging from 1 to 3

with

$a, b, c, \ldots$ ranging from 0 to 3

This is the convention used in d'Inverno's book.

Note that it is also common to use greek indices to represent spacetime indices. This has the advantage of making the distinction with the spatial indices in three dimensions more obvious.