MATH3006 Relativity, Black Holes & Cosmology

Coursework IV

This coursework is to be handed in to the Mathematics School Office by 12.00 on Wednesday 12 May 2010. The maximum total number of marks is 45, making up 10% of the final assessment for the course. Late work will be penalised in accordance with School policy (i.e. 10% each working day). You may discuss the problems with others, but you must write up the solutions independently.

The aim of this coursework is to show that weak gravity solutions to Einstein's field equations propagate as waves. You will be given some of the required results, but need to fill in a number of gaps in the derivation. The relations you need to derive are numbered!

Consider a metric that differs only slightly from the Minkowski metric. Let it be given by

$$g_{ab} = \eta_{ab} + \epsilon h_{ab} \ ,$$

where ϵ is a small (dimensionless) parameter. By carrying out calculations accurate to linear order in ϵ, i.e. neglecting all quadratic and higher order terms, one can show that

$$g^{ab} = \eta^{ab} - \epsilon h^{ab} \ .$$

Question 1. [4 marks] Verify that this results leads to

$$g^{ab} g_{bc} = \delta^a_c + O(\epsilon^2) \tag{1}$$

Moving on, note that at this level of approximation you can raise and lower indices with the flat metric. That is,

$$h^a{}_b = \eta^{ac} h_{cb}$$

From these results it follows that the Riemann tensor is given by

$$R_{abcd} = \frac{1}{2}\epsilon \left(h_{ad,bc} - h_{bd,ac} - h_{ac,bd} + h_{bc,ad}\right) \ ,$$

where a comma indicates a partial derivative.

Question 2. [10 marks] Show that this leads to

$$R_{ab} = -\frac{1}{2}\epsilon \left(\Box h_{ab} + h_{,ab} - h^c{}_{a,bc} - h^c{}_{b,ac}\right) \ , \tag{2}$$

where $h = h^a{}_a$ and $\Box = \eta^{ab}\partial_b\partial_a$, and

$$R = -\epsilon\left(\Box h - h^{cd}{}_{,cd}\right) \tag{3}$$

Next, one can show that a coordinate transformation of form

$$x^a \to x'^a = x^a + \epsilon\xi^a$$

affects the metric g_{ab} in such a way that

$$h_{ab} \to h'_{ab} = h_{ab} - \xi_{a,b} - \xi_{b,a}$$

Question 3. [6 marks] It turns out to be useful to define a new variable

$$\bar{h}_{ab} = h_{ab} - \frac{1}{2}\eta_{ab}h \ .$$

Show that this quantity transforms according to

$$\bar{h}_{ab} \to \bar{h}'_{ab} = \bar{h}_{ab} - \xi_{a,b} - \xi_{b,a} + \eta_{ab}\xi^c{}_{,c} \ . \tag{4}$$

Question 4. [15 marks] Use the previous results to show that the linearised Einstein equations in vacuum can be written

$$G_{ab} = R_{ab} - \frac{1}{2}\eta_{ab}R = -\frac{1}{2}\epsilon\left(\Box\bar{h}_{ab} + \eta_{ab}\bar{h}^{cd}{}_{,cd} - \bar{h}^c{}_{a,bc} - \bar{h}^c{}_{b,ac}\right) = 0 \ . \tag{5}$$

Question 5. [10 marks] Demonstrate that, if you choose gauge (=coordinates) such that

$$\bar{h}^a{}_{b,a} = 0 \ ,$$

then the problem simplifies to

$$G_{ab} = -\frac{1}{2}\epsilon\Box\bar{h}_{ab} = 0 \tag{6}$$

Finally, substitute in the original perturbation h_{ab} to show that

$$\Box h_{ab} = 0 \ . \tag{7}$$

This result shows that the linear metric perturbations satisfy a wave equation. This equation describes gravitational waves in weak gravity.