Energy momentum tensor

Newtonian gravity	General Relativity
gravitational potential $\Phi(x^i,t)$	metric g_{ab}
Newton's law $\frac{d^2x^i}{dt^2} = -\delta^{ij}\nabla_j\Phi$	Geodesic equation $\frac{d^2x^a}{d\tau^2} = -\Gamma^a_{bc}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau}$
Newtonian deviation $\frac{d^2\xi^i}{dt^2} = -\delta^{ij}\left(\frac{\partial^2\Phi}{\partial x^i\partial x^j}\right)\xi^k$	Geodesic deviation $\left(\nabla_{\mathbf{u}}\nabla_{\mathbf{u}}\xi\right)^a = -R^a{}_{bcd}u^b\xi^c u^d$
Tidal forces $\frac{\partial^2\Phi}{\partial x^i\partial x^j}$	Riemann curvature $R^a{}_{bcd}$
Laplace's equation $\nabla^2\Phi = 4\pi G\rho$	Einstein equations $G_{ab} = 8\pi G T_{ab}$

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SR dynamics

In order to discuss how we can construct suitable stress-energy tensors, we need to understand the role of energy and momentum in relativity better.

We already know that, in absence of forces, a body moves according to (in the local inertial frame)

$$\frac{du^a}{d\tau} = 0$$

When there are forces present, we can introduce an analogue of Newton's 2nd law by writing

$$m\frac{du^a}{d\tau} = f^a$$

This can't be "derived", but it satisfies key criteria;

– principle of relativity; it is in tensor form

– reduces to the force-free equation

– reduces to Newton's 2nd law at low velocities

Four acceleration

It now makes sense to introduce the four acceleration as

$$a^a = \frac{du^a}{d\tau} \quad \Rightarrow \quad f^a = ma^a$$

In a general frame this leads to $a^a = u^b \nabla_b u^a$.

It is worth noting that the normalisation of the four velocity means that we must have

$$\frac{d\left(u^a u_a\right)}{d\tau} = 0 \quad \Rightarrow \quad u_a a^a = 0 \quad \text{that is} \quad u_a f^a = 0$$

In other words, there are only three independent equations of motion – just like in Newtonian physics.

Four momentum

The equation of motion that we have written down leads naturally to the relativistic ideas of energy and momentum.

We define the four momentum as

$$p^a = mu^a \quad \Rightarrow \quad \frac{dp^a}{d\tau} = f^a$$

We also have

$$p^2 = p^a p_a = m^2$$

In a general inertial frame, moving with velocity v^i relative to the local inertial frame, we have

$$u^a = \left(\gamma, \gamma v^i\right) \quad \text{where} \quad \gamma = \left(1 - v^2\right)^{-1/2}$$

It then follows that, at low velocities, we have

$$p^t = \frac{m}{\sqrt{1-v^2}} \approx m + \tfrac{1}{2} m v^2 \quad \text{and} \quad p^i \approx m v^i$$

Natural to interpret $p^t = E$ as the energy and p^i as the three momentum.

Energy

Solving our relations for the energy, we arrive at

$$E = \left(m^2 + p^2\right)^{1/2}$$

which shows that the rest mass is part of the energy of a relativistic particle.

For a particle at rest, and in the usual units, we have

$$E = mc^2$$

This is, perhaps, the most famous equation in all of physics…

Note: It may be more appropriate to refer to p^a as the "energy-momentum" four vector.

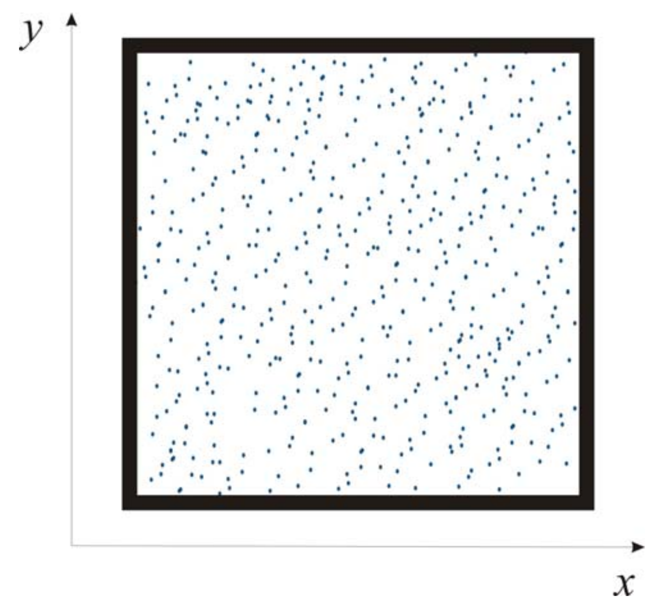
Also… it is the four momentum that is conserved in particle colliders like the LHC.

Number density

In order to build our intuition of the description of matter in relativity, it is useful to start by considering the number density of a gas.

Let us consider a box containing N particles. At rest, the volume of the box is V_*. Then the number density of particles in the box is simply

$$n = \frac{N}{V_*}$$



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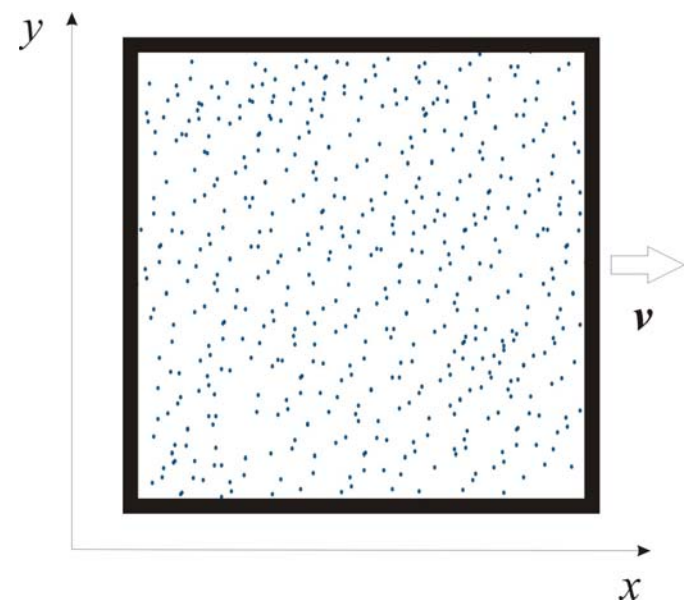
What happens if the box is moving?



Because of length contraction, the volume will be smaller;

$$V = \left(1 - v^2\right)^{1/2} V_*$$

but the total number of particles is the same, so the number density increases

$$\Rightarrow \quad \frac{n}{\sqrt{1-v^2}}$$

Number conservation

We see that the number density is nu^t. This suggests that we should introduce the particle flux four vector as

$$n^a = nu^a$$

This means that we have

$$n^a = \left(n^0, n^j\right) = \left(\frac{n}{\sqrt{1-v^2}}, \frac{nv^i}{\sqrt{1-v^2}}\right)$$

Using the argument that leads to the conservation of particles in fluid dynamics (flux through surface of some volume…) we can show that

$$\partial_t n^0 + \nabla \cdot \boldsymbol{n} = 0 \quad \Rightarrow \quad \nabla_a n^a = 0$$

Energy momentum tensor

We have seen how the number density current relates a scalar quantity with a volume. Let us now suppose that we want a similar argument for energy and momentum.

These are, however, given by the four-momentum. To relate this object with a volume, we need a tensor of rank 2.

This leads us to the energy-momentum tensor;

$$T^{ab} = \left(\begin{array}{c|c} \text{energy density} & \text{energy flux} \\ \hline \text{momentum density} & \text{stress tensor} \end{array} \right)$$

Consider the moving box, and assume that all particles are at rest with respect to the box. Then

$$\varepsilon = \text{energy density} = T^{00} = mnu^0u^0 = mn\gamma^2$$

$$\pi^i = \text{momentum density} = T^{0i} = mnu^0u^i = mn\gamma^2 v^i$$