Spacetime curvature
and
Einstein's equations

Riemann

So far, we have only discussed how we can take first derivatives of a tensor. We will now consider what happens if we need second derivatives.

It is quite easy to see that, in a curved spacetime, 2nd derivatives will not commute (recall the "untwisting" of the basis required to take the "limit" in the same tangent space).

However, since tensor calculus is linear, we should (quite generally) have

$$\nabla_c \nabla_d A^a - \nabla_d \nabla_c A^a = R^a{}_{bcd} A^b$$

This defines the Riemann tensor $R^a{}_{bcd}$ which provides a measure of the spacetime curvature.

From the definition we see that it must be anti-symmetric in the last two indices.

One can also prove (a bit harder) that it must be anti-symmetric in the first two indices and symmetric if the first and last pair of indices are interchanged.

Bianchi

In addition to the symmetries, the Riemann tensor must satisfy the so-called Bianchi identities. These are four relations, that can be written

$$\nabla_e R_{abcd} + \nabla_c R_{abde} + \nabla_d R_{abec} = 0$$

(cyclic permutation...)

Combining all this information one can work out that the Riemann tensor has 20 independent components.

They can be calculated from

$$R^a{}_{bcd} = \partial_c \Gamma^a{}_{bd} - \partial_d \Gamma^a{}_{bc} + \Gamma^a{}_{ce} \Gamma^e{}_{bd} - \Gamma^a{}_{de} \Gamma^e{}_{bc}$$

In other words, they follow from second partial derivatives of the metric g_{ab}.

Ricci

By contracting pairs of indices we obtain lower rank tensors.

Contract first and third index to get the Ricci tensor;

$$R^{c}{}_{bcd} = R_{bd} = R_{db}$$

Note that, due to the symmetries this is the only contraction of the Riemann tensor. Contracting on other indices can only give $\pm R_{bd}$.

Contract the remaining two indices to get the Ricci scalar;

$$R^{a}{}_{a} = R$$

Einstein

As an exercise, let us carry out the Ricci contraction on the Bianchi identities;

$$0 = g^{ac}\left(\nabla_e R_{abcd} + \nabla_c R_{abde} + \nabla_d R_{abec}\right) =$$

$$= \nabla_e R_{bd} + \nabla_c R^c{}_{bde} + \nabla_d R^c{}_{bec} = \nabla_e R_{bd} + \nabla_c R^c{}_{bde} - \nabla_d R_{be}$$

Another contraction leads to

$$0 = g^{be}\left(\nabla_e R_{bd} + \nabla_c R^c{}_{bde} - \nabla_d R_{be}\right) =$$

$$= \nabla^b R_{bd} + \nabla^c R_{cd} - \nabla_d R = 2\nabla^b R_{bd} - g_{db}\nabla^b R =$$

$$= \nabla^b\left(2R_{bd} - g_{db}R\right) = 2\nabla^b G_{bd}$$

This defines the Einstein tensor G_{ab}, which must satisfy the above differential constraint.

This object will be of great importance later. In fact, we will find that

$$G_{ab} = R_{ab} - \tfrac{1}{2} g_{ab} R = 0$$

are the Einstein field equations in vacuum.

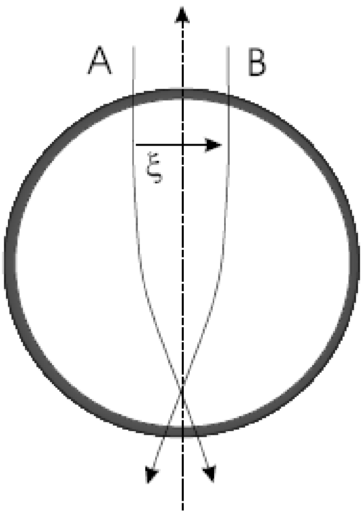
Geodesic deviation

Given a description of the curvature, we can now consider the relative changes in two neighbouring geodesics.

To do this, let us assume that the geodesics are separated by a (spatial) vector ξ^a, such that

$$u_a \xi^a = 0$$

Our aim is to understand how ξ^a is affected by the spacetime curvature.



Use the proper time τ as parameter for the geodesics and label them by a parameter λ, such that

$$u^a = \left(\frac{\partial x^a}{\partial \tau}\right)_\lambda \qquad \xi^a = \left(\frac{\partial x^a}{\partial \lambda}\right)_\tau$$

then

$$\nabla_{\boldsymbol{u}} \xi - \nabla_\xi \boldsymbol{u} = u^b \nabla_b \xi^a - \xi^b \nabla_b u^a = 0$$

This leads to

$$\nabla_{\boldsymbol{u}} \nabla_{\boldsymbol{u}} \xi = \nabla_{\boldsymbol{u}} \nabla_\xi \boldsymbol{u} = \left(\nabla_{\boldsymbol{u}} \nabla_\xi - \nabla_\xi \nabla_{\boldsymbol{u}}\right) \boldsymbol{u}$$

or, in component form;

$$u^c \nabla_c \left(u^a \nabla_a \xi^b\right) = \left(\nabla_a \nabla_c u^b - \nabla_c \nabla_a u^b\right) \xi^a u^c$$

The relative acceleration is caused by the failure of second derivatives to commute. We have the equation for geodesic deviation;

$$u^c \nabla_c \left(u^a \nabla_a \xi^b\right) = -R^b{}_{dac} u^d \xi^a u^c$$

Spacetime curvature

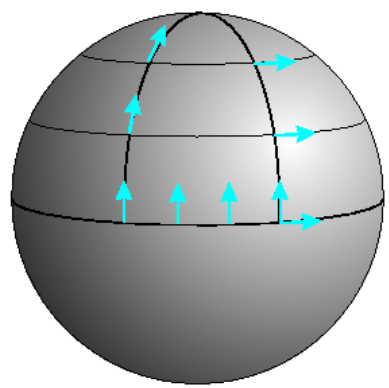
We are aiming to link the spacetime curvature to gravity. In order to do this we need to understand what we mean by “curvature”.

Distinguish between intrinsic and extrinsic curvature.

Extrinsic curvature considers the surface in a higher dimensional flat space.

Intrinsic curvature measures to what extent parallel transported vectors remain parallel:

flat



curved

In a curved spacetime the result of parallel propagation depends on the path.

Parallel lines do not remain parallel when extended (cf. great circles on the sphere).

Meanwhile, in a flat spacetime parallelism is global.

Relating this to geodesics, we see that geodesics in flat space maintain their separation, while those in curved spaces don't.

From the equation for geodesic deviation;

$$u^c \nabla_c \left(u^a \nabla_a \xi^b \right) = -R^b{}_{dac} u^d \xi^a u^c$$

We then see that the Riemann tensor provides a measure of the curvature.

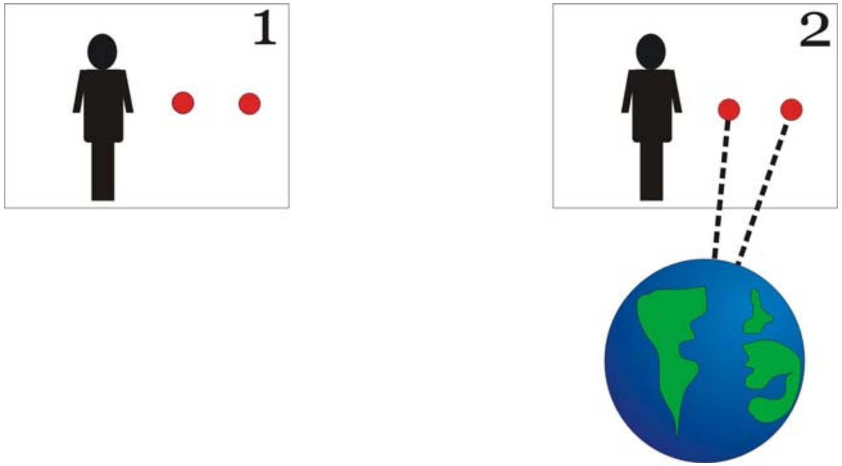
In a flat spacetime $R^a{}_{bcd}$ must vanish.

<u>Note</u>: A spacetime is flat if all $\Gamma^a{}_{bc}$, and their derivatives, vanish.

In a curved spacetime we can make $\Gamma^a{}_{bc}$ and the first derivatives vanish, but the second derivatives will be non-zero.

Lift experiments (again)

How do we “measure” the spacetime curvature?



Idea: There should be a link with geodesic deviation.

The tidal force of the gravitational field can be represented by the spacetime curvature.

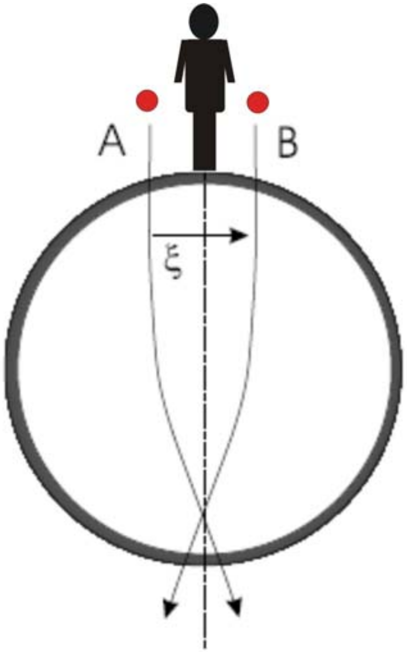
Newtonian problem

Let a person standing on the surface of the Earth drop two objects in such a way that they initially fall along parallel trajectories. Assume that the two objects are “ideal" in the sense that their motion is only affected by gravity.

We know that gravity attracts towards the centre of the Earth so the trajectories of the two objects should eventually cross.

In Newtonian physics, this happens because of the universal gravitational attraction.

Let us describe this problem mathematically.



Taking the separation vector to be ξ^i we must have, since the trajectories start out parallel;

$$\frac{d\xi^i}{dt} = 0$$

The second derivatives will not vanish so, in general we should have

$$\frac{d^2\xi^i}{dt^2} = -E^i{}_j \xi^j$$

This defines the tidal tensor ***E***. We need to relate it to the gravitational potential Φ. We know that the acceleration should be proportional to the gradient of the potential. That is,

$$\frac{d^2x^i}{dt^2} = -\nabla^i \Phi$$

The separation vector is defined as $\xi^i = x^i_A - x^i_B$ so (Taylor expanding) we find that

$$\frac{d^2\xi^i}{dt^2} = -\delta^{ik}\left(\frac{\partial^2\Phi}{\partial x^k \partial x^j}\right)\xi^j \quad \Rightarrow \; E^i{}_j = \delta^{ik} E_{kj} = \delta^{ik}\frac{\partial^2\Phi}{\partial x^k \partial x^j}$$

Once we know the gravitational potential, i.e., the mass distribution, we can work out the relative motion of the two particles.

Spacetime description

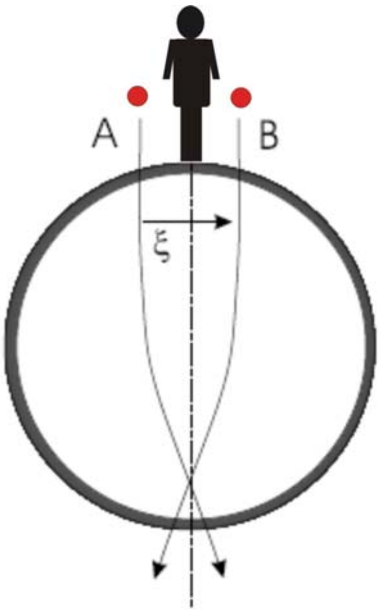
In a curved spacetime, the problem works out quite differently...

We obviously need to consider the equation for geodesic deviation. Hence, we take as our starting point

$$u^c \nabla_c \left(u^a \nabla_a \xi^b \right) = -R^b{}_{dac} u^d \xi^a u^c$$

Opting to work in the local inertial frame of A, we have

$$u^0 = 1 \qquad u^j = 0 \qquad \xi^0 = 0 \qquad \xi^j \neq 0$$



and it follows that

$$\left(\nabla_{\boldsymbol{u}} \nabla_{\boldsymbol{u}} \xi \right)^j = \frac{\partial^2 \xi^j}{\partial t^2} = -R^j{}_{abc} u^a \xi^b u^c = -R^j{}_{0b0} \xi^b = -R^j{}_{0k0} \xi^k$$

Comparing to the Newtonian result, we identify

$$R_{j0k0} = E_{jk}$$

We now have a contraint on the curved spacetime theory (Newtonian correspondence). However, we need to make the model covariant...

From Newtonian theory, we also know that

$$4\pi G\rho = \nabla^2\Phi = \delta^{jk}\nabla_j\nabla_k\Phi = \delta^{jk}E_{jk} = E^j{}_j$$

This suggests that we want our field equations to look something like

$$R^j{}_{0j0} = 4\pi G\rho$$

But this is not covariant – we should have spacetime indices. Hence, we could use

$$R_{00} = R^a{}_{0a0} = 4\pi G\rho$$

Inspired by this, Einstein suggested that the field equations of relativity should be

$$R_{ab} = 4\pi G T_{ab}$$

where the stress-energy tensor T_{ab} (more later) is such that (weak fields)

$$T_{00} \approx \rho$$

This suggestion satifies many of the requirements... yet it still fails.

A closer look shows that

$$R_{ab} = 4\pi G T_{ab}$$

provides 10 equations for the 10 unknown metric coefficients. Just what we would want... or is it?

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A closer look shows that

$$R_{a\ldots} \ldots {}_{ab}$$

provid… tions for the 10 unknown metric coefficients. Just what we would want... or is it?

In fact, it is not.

The geometric description allows us to freely choose the coordinate system. In principle, We can use this freedom to make four of the metric functions anything we want. The problem is overdetermined.

We need to formulate the field equations in such a way that we have six independent equations.

With 100 years of hindsight...

... the solution is simple.

We should use

$$\boxed{G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R = 8\pi G T_{ab}}$$

The contracted Bianchi identities provide the four constraints

$$\nabla^a G_{ab} = 0$$

Which implies that we must also have

$$\nabla^a T_{ab} = 0$$

As we will see later, this implies the conservation of energy and momentum.

"Space tells matter how to move and matter tells space how to curve."

John Wheeler

Special relativity

I. Space and time are represented by a four dimensional manifold with symmetric connection Γ^a_{bc} and metric tensor satisfying:

a) the metric g_{ab} is non-singular (with signature -2)

b) $\nabla_a g_{bc} = 0$

c) $R^a{}_{bcd} = 0$

II. There are privileged classes of curves;

a) ideal clocks travel along timelike curves and measure “proper” time

b) free particles travel along timelike geodesics

c) light rays travel along null geodesics

General relativity

I. Space and time are represented by a four dimensional manifold with symmetric connection Γ^{a}_{bc} and metric tensor satisfying:

a) the metric g_{ab} is non-singular (with signature -2)

b) $\nabla_a g_{bc} = 0$

c) $R \quad 0 \quad \Rightarrow \quad G_{ab} = 8\pi G T_{ab}$

II. There are privileged classes of curves;

a) ideal clocks travel along timelike curves and measure “proper” time

b) free particles travel along timelike geodesics

c) light rays travel along null geodesics