Tensor calculus and geodesics

MATH3006
Relativity, black holes and cosmology

Tensors

At this point it is natural to return to the question of what a tensor actually is...

There are different approaches to this. The most "elegant" solution is to focus on the geometric interpretation. Then a tensor is...

... a linear function of a number of vectors. Each tensor has a number of " slots" that can be filled with vectors. The number of slots determines the rank of the tensor. If all slots are filled the result is a number, if one slot is left empty the outcome is a vector, and so on.

This picture is elegant, but it does not really tell us how to calculate with tensors. A more "practical" start may be to focus on a few examples.

Contravariant vectors

Let us consider the displacement vector dx^a that connects two events with coordinates x^a and $x^a + dx^a$.

Suppose we want to work out how this vector changes if we make the coordinate change $x^a \rightarrow x'^a$

Thinking of the new (primed) coordinates as functions of the old (unprimed) coordinates, we see that the required transformation is

$$dx'^a = \frac{\partial x'^a}{\partial x^b} dx^b$$

This transformation property defines a (contravariant) vector. In general, we have

$$X'^a = \frac{\partial x'^a}{\partial x^b} X^b$$

A typical example is the tangent vector to a curve (e.g. the four velocity)

$$x^a = x^a(\tau) \quad \rightarrow \quad u^a = \frac{dx^a}{d\tau}$$

Covariant vectors

In this case it is natural to start with a scalar field $\phi=\phi(x^a)$.

Again consider the coordinate change $x^a \to x'^a$ but now think of the old (unprimed) coordinates as functions of the new (primed) coordinates.

It follows that

$$\phi = \phi(x^a(x')) \quad \to \quad \frac{\partial \phi}{\partial x'^a} = \frac{\partial x^b}{\partial x'^a}\frac{\partial \phi}{\partial x^b}$$

This transformation (which involves the inverse transformation matrix) property defines a covariant vector. In general, we have

$$X'_a = \frac{\partial x^b}{\partial x'^a} X_b$$

The archetypal example is the gradient.

Partial derivatives

When we are dealing with tensors, we can define different kinds of derivatives. The main requirement is that the operation is tensorial.

Let us first consider the (usual) partial derivative;

$$\frac{\partial X^a}{\partial x^b} \qquad \partial_b X^a \qquad X^a{}_{,b}$$

How does this object transform if we change coordinates? After some work, we find that

$$\partial'_c X'^a = \frac{\partial}{\partial x'^c}\left(\frac{\partial x'^a}{\partial x^b} X^b\right) = \frac{\partial x^d}{\partial x'^c}\frac{\partial}{\partial x^d}\left(\frac{\partial x'^a}{\partial x^b} X^b\right)$$

$$= \frac{\partial x^d}{\partial x'^c}\frac{\partial x'^a}{\partial x^b}\partial_d X^b + \frac{\partial^2 x'^a}{\partial x^d \partial x^b}\frac{\partial x^d}{\partial x'^c} X^b$$

So... the partial derivative does not transform like a tensor. Hence, it is not (without modification) a useful concept for tensors.

Covariant derivative

It is quite easy to see why we ran into trouble with the partial derivative. Recall that a vector, like the four velocity, is generally written

$$\boldsymbol{u} = u^a \boldsymbol{e}_a$$

If we want to define the rate of change as

$$\nabla \boldsymbol{u} = \lim_{\Delta\tau \to 0} \frac{\boldsymbol{u}(\tau + \Delta\tau) - \boldsymbol{u}(\tau)}{\Delta\tau}$$

Then we need to account for the fact that the two four velocities "live" in different tangent spaces. In order to make the operation meaningful, we need to move one of the vectors so that the limit can be taken in <u>the same</u> tangent space.

This sounds complicated...

... but, in fact, it is not so bad.

We need to work out the stretching and twisting of the basis as we move through spacetime. This requires the connection between the two tangent spaces.

Suppose we want to define the derivative $\nabla = \nabla_a = \nabla_{\boldsymbol{e}_a}$

First of all, it is easy to see that we have no problem with scalars. In that case there is no issue with tangent spaces, and it follows that

$$\boxed{\nabla\phi = \nabla_a\phi = \partial_a\phi}$$

In the case of a vector, like the four velocity, we need

$$\nabla_a\boldsymbol{u} = \nabla_a\left(u^b\boldsymbol{e}_b\right) = \left(\nabla_a u^b\right)\boldsymbol{e}_b + u^b\left(\nabla_a\boldsymbol{e}_b\right) = \left(\partial_a u^b\right)\boldsymbol{e}_b + u^b\left(\nabla_a\boldsymbol{e}_b\right)$$

We now make use of the fact that tensor calculus is linear. We should be able to write any expression as a linear combination of the basis vectors. Hence,

$$\nabla_a\boldsymbol{e}_b = \Gamma^c_{ba}\boldsymbol{e}_c$$

which leads to the covariant derivative;

$$\nabla_a\boldsymbol{u} = \left(\partial_a u^b\right)\boldsymbol{e}_b + u^b\Gamma^c_{ba}\boldsymbol{e}_c = \left(\partial_a u^b + \Gamma^b_{ca}u^c\right)\boldsymbol{e}_b$$

$$\boxed{\nabla_a u^b = u^b{}_{;a} = \partial_a u^b + \Gamma^b_{ca}u^c}$$

Using the definition of the dual basis $\boldsymbol{e}^a \cdot \boldsymbol{e}_b = \delta^a_b$ one can also show that

$$\nabla_a \boldsymbol{e}^b = -\Gamma^b_{ca} \boldsymbol{e}^c$$

For a covector we then get

$$\boxed{\nabla_a u_b = u_{b;a} = \partial_a u_b - \Gamma^c_{ba} u_c}$$

<u>A few comments:</u>

i) The (affine) connection Γ^c_{ab} does not transform as a tensor.

ii) We will assume that spacetime is torsion free, which means that the connection is symmetric;

$$\Gamma^c_{ab} = \Gamma^c_{ba}$$

iii) The connection vanishes in Minkowski space.

Christoffel symbols

We (obviously) want to be able to work out the connection for a given spacetime.

To do this, we require that the covariant derivative is "compatible" with the metric. This means that we want the derivative to be such that

$$\nabla_a g_{bc} = 0$$

Working this out, we see that we should have

$$\nabla_c g_{ab} = \partial_c g_{ab} - \Gamma^d_{ac} g_{db} - \Gamma^d_{bc} g_{ad} = 0$$

This can be used to show that the metric connection is

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad} \left(\partial_b g_{dc} + \partial_c g_{db} - \partial_d g_{bc} \right)$$

These coefficients are also known as the Christoffel symbols.

Note: We also have

$$\nabla_c \delta^a_b = 0 \quad \rightarrow \quad \nabla_c g^{ab} = 0$$

Directional derivative

We want to be able to distinguish "freely falling" objects, i.e. bodies that are not acted on by any "forces". It is easy to define this concept in the local inertial frame.

The four-velocity should be constant, such that

$$u^a = \frac{dx^a}{d\tau} = k^a = \text{constant} \quad \rightarrow \quad x^a = k^a \tau + l^a$$

This means that the world-line is "as straight as possible".

Another way to obtain the result is to demand that the derivative of u^a along itself must vanish. Thus we define the directional derivative as

$$\nabla_{\boldsymbol{u}} u^a \equiv u^b \nabla_b u^a$$

A vector is parallel transported if the absolute derivative

$$\frac{Du^a}{D\tau} = \nabla_{\boldsymbol{u}} u^a = 0$$

Affine geodesics

An affine geodesic is a curve along which the tangent vector is parallel propagated.

This means that the four-velocity should be constant, such that

$$\frac{Du^a}{D\tau} = \lambda(\tau)u^a$$

Alternatively, this can be written

$$\nabla_{\boldsymbol{u}} u^a = \lambda u^a$$

or

$$\frac{d^2x^a}{d\tau^2} + \Gamma^a_{bc}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} = \lambda\frac{dx^a}{d\tau}$$

Metric geodesics

If the curve is parametrized in such a way that λ vanishes, then we are using an affine parameter. For a timelike worldline we can use the proper time.

This leads to the equation for metric geodesics

$$\nabla_{\boldsymbol{u}} u^a = u^b \nabla_b u^a = 0$$

or

$$\frac{d^2 x^a}{d\tau^2} + \Gamma^a_{bc} \frac{dx^b}{d\tau} \frac{dx^c}{d\tau} = 0$$

where

$$g_{ab} \frac{dx^a}{d\tau} \frac{dx^b}{d\tau} = 1$$

Null geodesics

There will also exist geodesics for which the distance between different points is zero.

To describe such null geodesics, we need some other affine parameter, *s*, say. Then we have

$$\frac{d^2x^a}{ds^2} + \Gamma^a_{bc}\frac{dx^b}{ds}\frac{dx^c}{ds} = 0$$

with

$$g_{ab}\frac{dx^a}{ds}\frac{dx^b}{ds} = 0$$

By studying the geodesics of a given spacetime we learn how both massive objects and light move.