The spacetime view

Spacetime events

In developing relativity, we need to think carefully about many concepts that we would normally take for granted.

We have already been forced to abandon the notion of "simultaneity".

We want the mathematical model to be coordinate independent.

It is natural to think of spacetime as being made up of events, P, which can be defined without resorting to a given coordinate system.

Of course, given a set of coordinates x^a $(a=0\text{-}3)$ there is a one-to-one mapping between the events and particular coordinate values.

The spacetime interval

Consider two events P and Q, and define the vector $\Delta \boldsymbol{x}$ that separates them. It is easy to see that, even though different observers may disagree on the coordinate values associated with the events, they will agree both on the events and the vector that separates them.

The principle of relativity forces the interval (the “distance” in spacetime)

$$ds^2 = \left(\Delta s\right)^2 = \left(\Delta \boldsymbol{x}\right)^2$$

to be the same in all reference frames.

This means that we can associate the squared length of a vector with the spacetime interval.

However, we have already seen that the scalar product (which we need to work out the length of a vector) requires the use of the metric. It follows that

$$ds^2 = g_{ab} dx^a dx^b = (d\boldsymbol{x})^2$$

Lorentz transformation

We can generalize the Lorentz transformation to describe a boost in any direction. To do this, we combine the idea with an ordinary rotation.

Mathematically, a <u>rotation</u> is a transformation in 3 dimensions which leaves the distance between two points invariant;

$$\sigma^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$$

A <u>Lorentz</u> transformation leaves the notion of an <u>interval</u> is spacetime invariant;

$$s^2 = (t_1 - t_2)^2 - (x_1 - x_2)^2 - (y_1 - y_2)^2 - (z_1 - z_2)^2$$

The infinitesimal version of this is

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2$$

Minkowski metric

Let us consider the case of a flat spacetime, described in terms of the usual Cartesian coordinates $x^a=\{t,x,y,z\}$. The corresponding line element takes the simple form

$$ds^2 = g_{ab}dx^a dx^b = dt^2 - dx^2 - dy^2 - dz^2$$

(signature -2).

We have already seen that this interval is invariant under the Lorentz transformation. In fact, this metric (known as the Minkowski spacetime) is the metric associated with special relativity. It covers the whole manifold.

The Minkowski metric is usually written $g_{ab} = \eta_{ab}$ where

$$\eta_{ab} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Inertial frames

In order to discuss motion in spacetime, we need to develop the concept of an inertial frame.

It is natural to think of an inertial frame as a set of rods and clocks that moves in such a way that it is not affected by any forces.

The rods form an orthogonal lattice with uniform length intervals that can be used to set up a (local) Cartesian coordinate system. The clocks are synchronized via light signals.

Once we identify an inertial frame, we have a natural coordinate system to record events. The coordinates associated with an event P is simply the coordinate location $\{x,y,z\}$ of the event and the time t measured by the clock at that location.

In a curved spacetime one cannot construct a global inertial frame, but the concept is still useful locally. Such local inertial frames are relevant in a small region of spacetime.

The world line

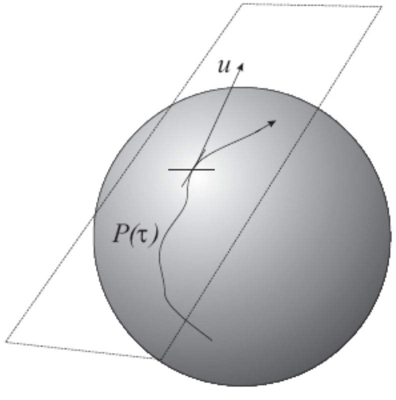
Now consider a moving body. The world-line of the body is the sequence of events $P(\tau)$, where τ is the (proper) time recorded by the ideal clock carried by the moving observer.

The tangent vector to the world-line defines the four velocity u^a.

We define the four velocity using the standard "limiting" procedure;

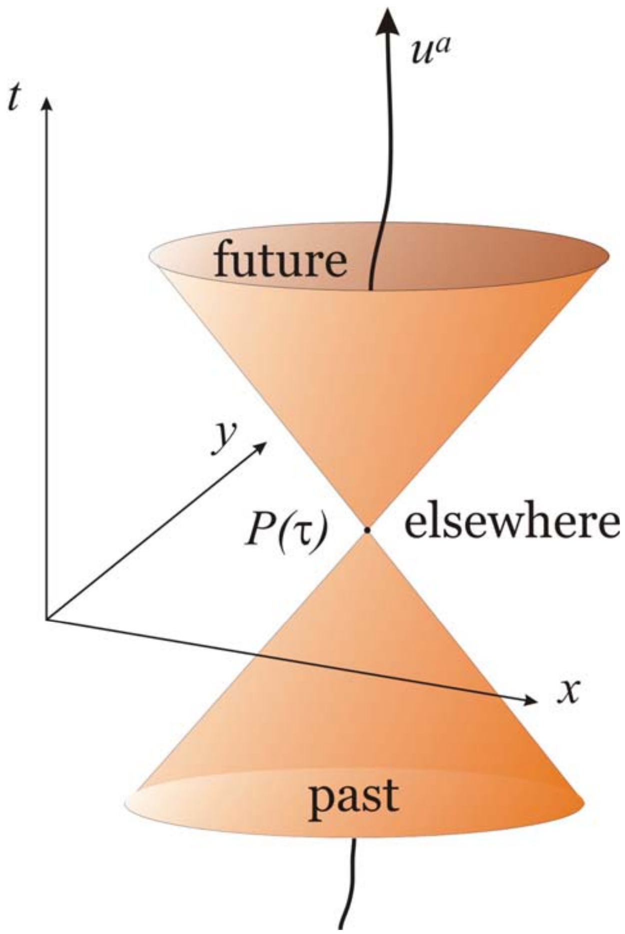
$$u^a = \frac{dx^a}{d\tau} = \frac{dP}{d\tau} = \lim_{\Delta\tau \to 0} \frac{P(\tau + \Delta\tau) - P(\tau)}{\Delta\tau}$$

Note: in a curved spacetime the tangent vector lives in the flat tangent space associated with each spacetime point P.



The light cone

The light-cone defines the past and the future of each spacetime event.



Given the (squared) norm of any spacetime vector

$$X^2 = g_{ab} X^a X^b = X_a X^a$$

We can distinguish three cases

$X^2 > 0$ timelike

$X^2 = 0$ null (lightlike)

$X^2 < 0$ spacelike

Since the speed of light sets the speed limit, only events whose separation is timelike can be a part of the future (past) of the event P.

Proper time

The time measured by the observers ideal clock (obviously) represents a "timelike" interval;

$$ds^2 = d\tau^2 > 0$$

This defines the "proper time". Since the spacetime interval is invariant, it follows immediately that

$$u^a u_a = \frac{d\mathbf{x}}{d\tau} \cdot \frac{d\mathbf{x}}{d\tau} = \eta_{ab} \frac{dx^a}{d\tau} \frac{dx^b}{d\tau} = \frac{ds^2}{d\tau^2} = 1$$

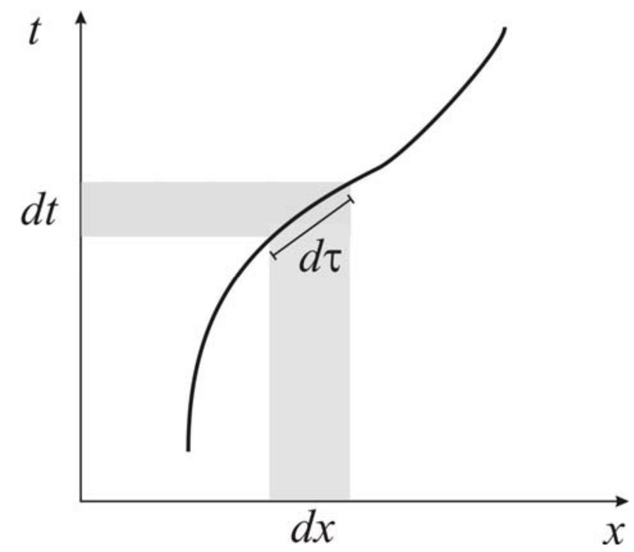
That is, the four velocity is always normalized to unity.

The four velocity is a timelike vector.

Time dilation (again)

As an example, let us now consider how the proper time τ relates to the coordinate time *t* for an observer that moves with velocity

$$\boldsymbol{v} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$$



In this case we have

$$\begin{aligned} d\tau^2 &= ds^2 = dt^2 - dx^2 - dy^2 - dz^2 \\ &= dt^2 \left\{ 1 - \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 \right] \right\} \\ &= dt^2 \left(1 - v^2\right) \end{aligned}$$

If the velocity is constant, this is the time dilation result that we had before;

$$dt = \frac{d\tau}{\left(1 - v^2\right)^{1/2}}$$