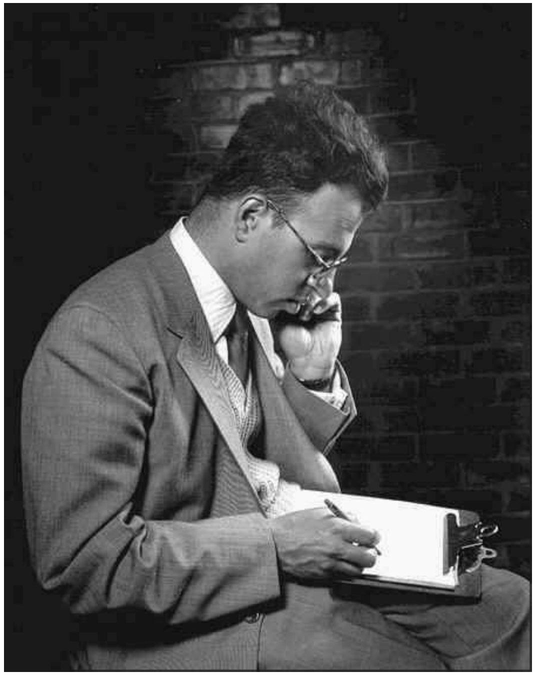
Bondi's k-calculus

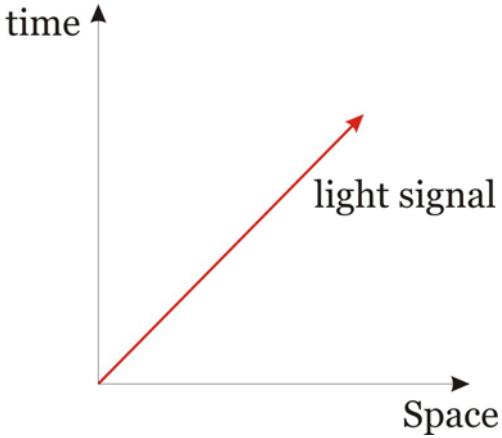
k-calculus

In order to derive the key results of Special Relativity, we will make use of the k-calculus, invented by Hermann Bondi.



The key idea is to use light signals to relate measurements carried out by observers in (non-accelerated) relative motion.

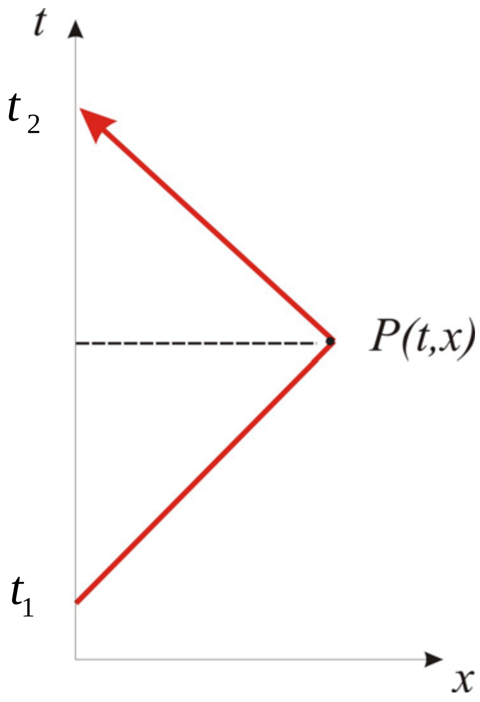
If c=1 then light-signals travel at 45^o in a "space-time" diagram.



Radar method

Idea: Use the constancy of the speed of light to measure distance.

(e.g. time in seconds, distance in light-seconds)



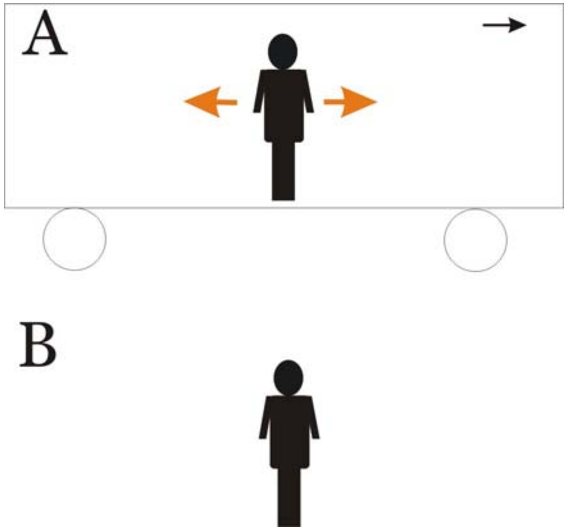
From the diagram it is easy to see that

$$t = \frac{1}{2}\left(t_1 + t_2\right)$$

$$x = \frac{1}{2}\left(t_2 - t_1\right)$$

Simultaneity

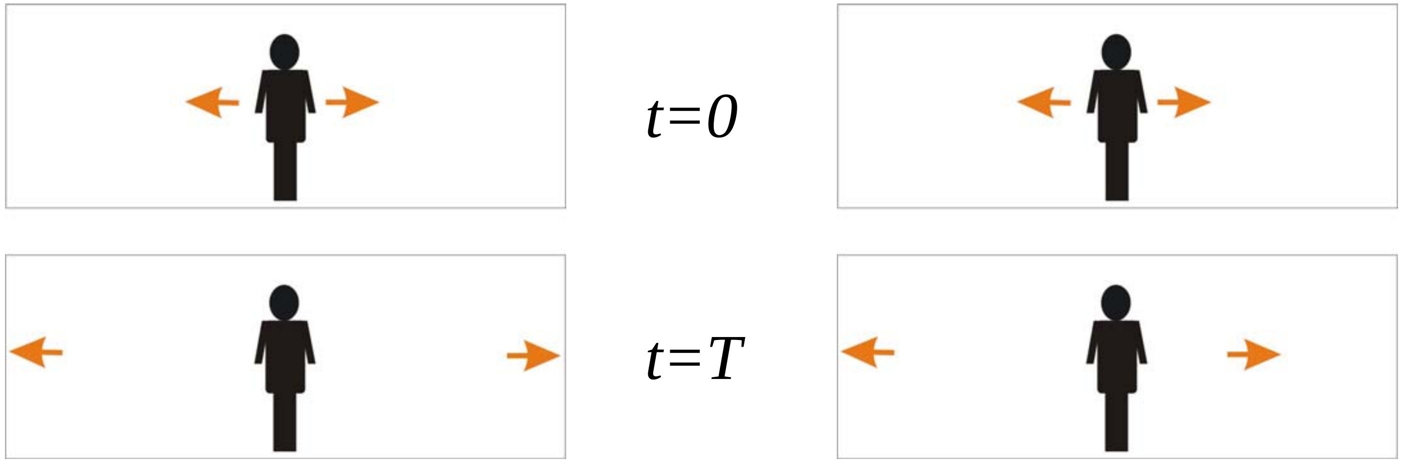
Consider a simple experiment as viewed by two observers in relative motion.



Simultaneity

Consider a simple experiment as viewed by two observers in relative motion.

According to A: According to B:



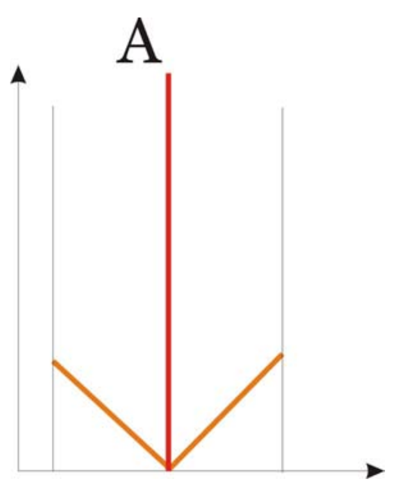
photons arrive at the same time — photon arrives at back before the front

Lesson: Time coordinate of an "event" depends on the observer.

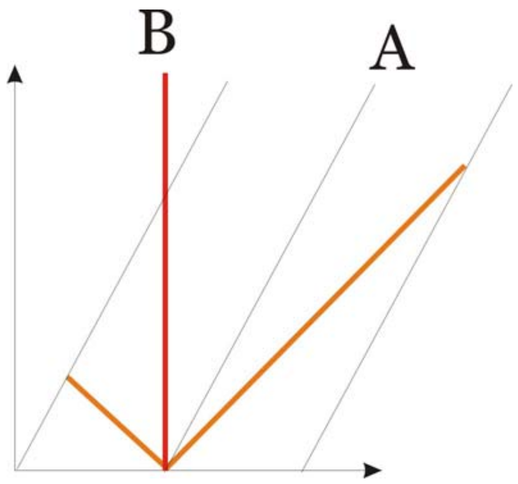
Simultaneity

Consider a simple experiment as viewed by two observers in relative motion.

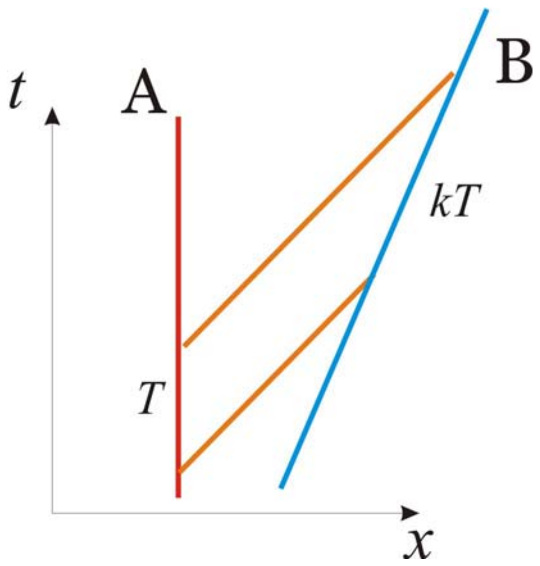
According to A:

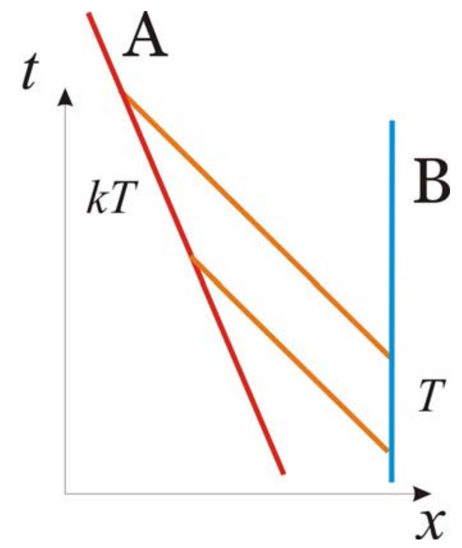


According to B:



The k-factor





Let us now assume that if A observes a time interval T (say), between two light pulses, then B observes an interval kT.

Conversely;

If B observes a time interval T then A observes an interval kT.

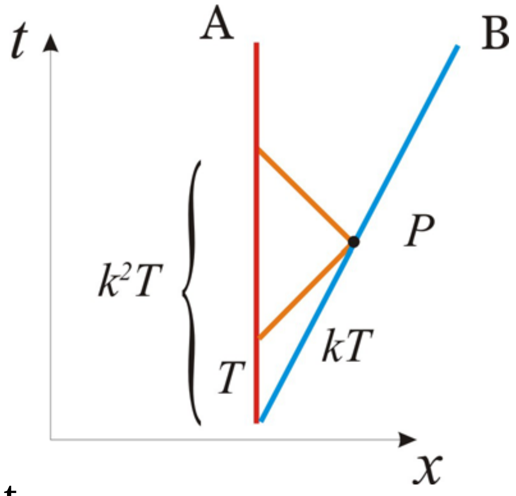
Relative speed

What are the coordinates of an event P?

The "radar method" leads to;

$$t = \frac{1}{2}\left(T + k^2 T\right) = \frac{1}{2}\left(k^2 + 1\right)T$$

$$x = \frac{1}{2}\left(k^2 T - T\right) = \frac{1}{2}\left(k^2 - 1\right)T$$

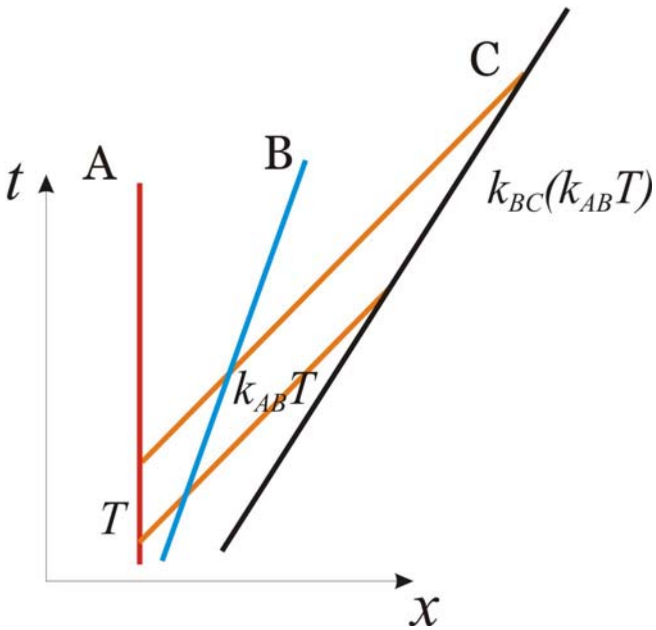


If B has speed v relative to A, it follows that

$$v = \frac{x}{t} = \frac{k^2 - 1}{k^2 + 1} \quad \Rightarrow \quad k = \left(\frac{1+v}{1-v}\right)^{1/2}$$

Adding velocities

Let us now consider three observer moving relative to one another.



We find that

$$k_{AC} = k_{AB}k_{BC} \quad \Rightarrow \quad v_{AC} = \frac{v_{AB} + v_{BC}}{1 + v_{AB}v_{BC}}$$

That is, velocities do not add in the way that we are used to from Newtonian theory.

"Verified" by Fizeau's experiment (1851).

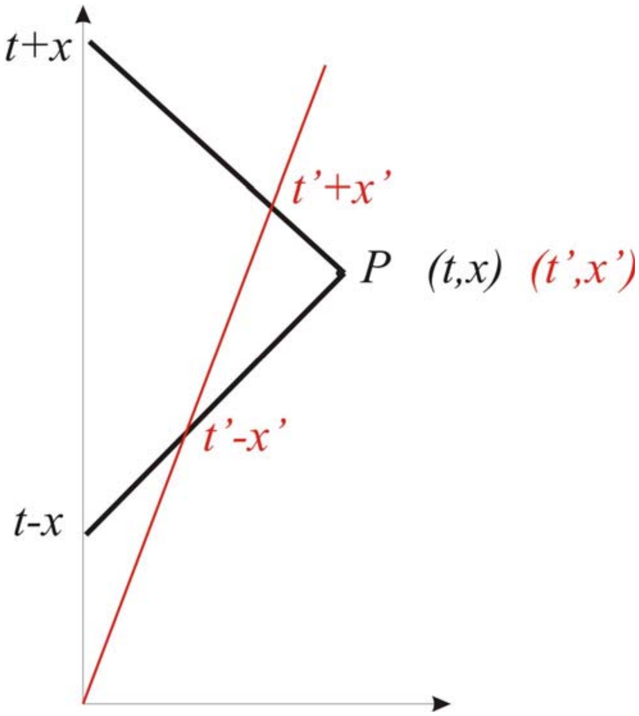
<u>Two limits:</u>

$$v_{AB}, v_{BC} \ll 1 \quad \Rightarrow \quad v_{AC} \approx v_{AB} + v_{BC}$$

$$v_{AB}, v_{BC} \to 1 \quad \Rightarrow \quad v_{AC} \to 1$$

Lorentz transformation

Now consider an event *P* viewed by two different observers.



$$t' - x' = k(t - x)$$

$$t + x = k(t' + x')$$

$$t' = \frac{1}{2}\left[\frac{1}{k}(t+x) + k(t-x)\right] = \frac{t - vx}{(1-v^2)^{1/2}}$$

$$x' = \frac{1}{2}\left[\frac{1}{k}(t+x) - k(t-x)\right] = \frac{x - vt}{(1-v^2)^{1/2}}$$

For low velocities we have

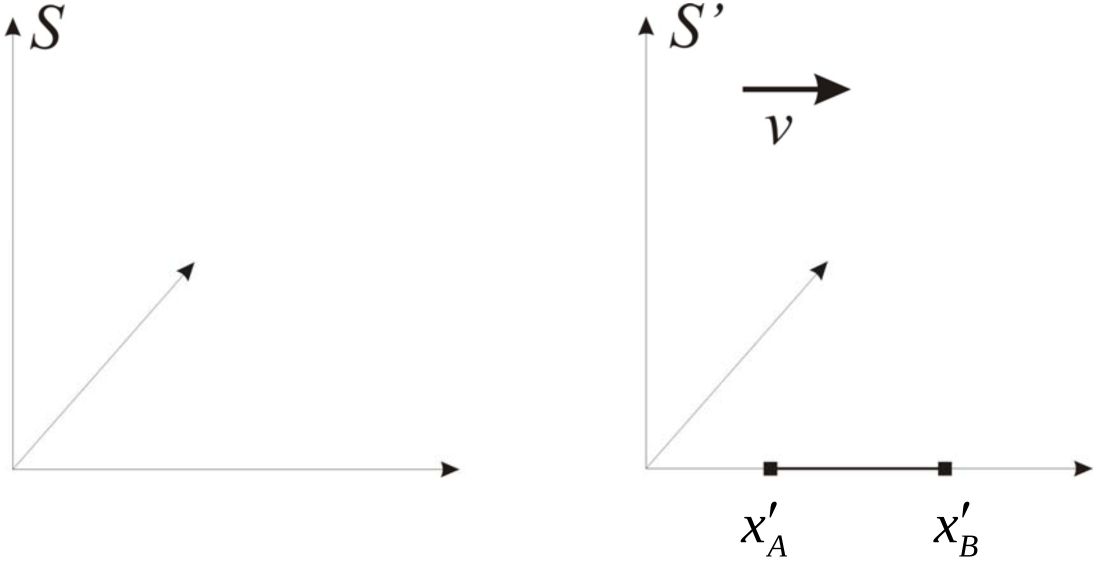
$$t' \approx t \quad \text{and} \quad x' \approx x - vt$$

i.e., a <u>boost</u> in the *x*-direction.

Length contraction

Consider a rod moving with speed *v* in the *x*-direction.

Let *S'* move with the same velocity so that the rod is at rest in this frame.



The <u>rest length</u> (proper length) of the rod is $l_0 = x'_B - x'_A$

How long does the rod appear to be to observer *S*?

Length contraction

Measure x_A and x_B at the same time t_0:

$$l = x_B - x_A$$

But we know that

$$x'_A = \beta\left(x_A - vt_0\right) \quad \text{where} \quad \beta = \left(1 - v^2\right)^{-1/2}$$

$$x'_B = \beta\left(x_B - vt_0\right)$$

So we find

$$x'_B - x'_A = \beta\left(x_B - x_A\right) \quad \Rightarrow \quad l_0 = \beta l \quad \Rightarrow \quad l < l_0$$

This is an example of length contraction, a moving object appears "shorter" than it is at rest.

Time dilation

Carry out the same exercise for a clock moving with velocity *v*. Let the clock be at rest in the (moving) *S'* frame.

Assume that the interval between ticks on the clock is T_0 (proper time). Then ticks correspond to events

$$\left(x_1', t_1'\right) \quad \text{and} \quad \left(x_1', t_1' + T_0\right)$$

Transforming to the non-moving frame, *S*, we have

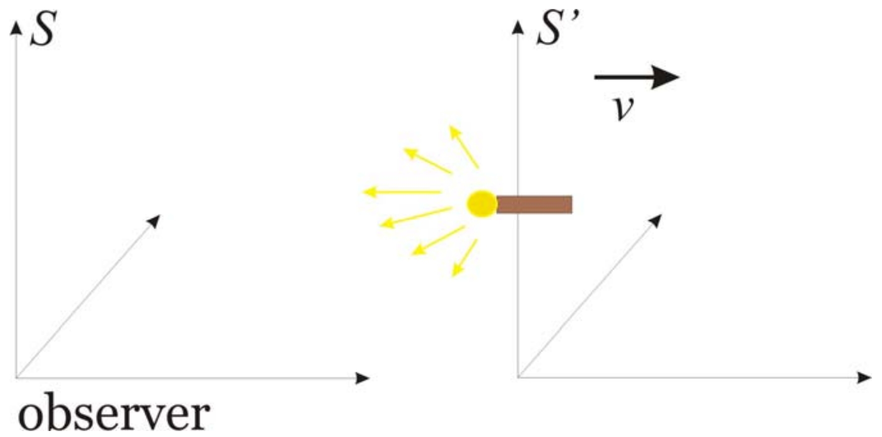
$$\left.\begin{aligned} t_1 &= \beta\left(t_1' + v x_1'\right) \\ t_2 &= \beta\left(t_1' + T_0 + v x_1'\right) \end{aligned}\right\} \quad \Rightarrow \quad T = t_2 - t_1 = \beta T_0 \quad \Rightarrow \quad T > T_0$$

So the moving clock appears to run slower. This is called "time dilation".

Doppler effect

Let us first consider the classical (non-relativistic) Doppler effect.

Consider a light source whose wavelength in the rest frame is λ_0. Suppose this source is moving away from an observer with speed v.



If two photons are emitted a time *dt'* apart, then the second photon has to travel an extra distance *vdt'*, and hence arrive *vdt'/c* later. That is, the photons arrive with a time difference (measured by *S'*)

$$\Delta t' = \left(1 + \frac{v}{c}\right)\delta t' \quad \Rightarrow \quad \frac{\lambda}{\lambda_0} = 1 + \frac{v}{c}$$

Relativistic Doppler shift

Now consider the same problem from the point of view of special relativity.

Due to time dilation, the time difference measured by S is

$$\delta t = \beta \Delta t' = \beta \left(1 + \frac{v}{c} \right) \delta t'$$

Since the observed wavelengths are

$$\lambda = c\delta t \quad \text{and} \quad \lambda_0 = c\delta t'$$

we see that

$$\lambda = \beta \left(1 + \frac{v}{c} \right) \lambda_0 \quad \text{or} \quad \frac{\lambda}{\lambda_0} = \frac{1 + v/c}{\left(1 - v^2/c^2\right)^{1/2}}$$

The observed wavelength is longer, the light has been red-shifted.

Note that,

$$\frac{\lambda}{\lambda_0} = \frac{(1 + v/c)^{1/2}}{(1 - v/c)^{1/2}} = \text{ the k-factor}$$