A few steps towards relativity: Linear plane waves

MATH3006
Relativity, black holes and cosmology

The speed of sound

Let us consider the propagation of sound waves in a moving fluid. That is, we consider the Euler equations

$$\partial_t \rho + \nabla_i (\rho v^i) = 0$$

$$\partial_t v^i + (v^j \nabla_j) v^i + \frac{1}{\rho} \nabla^i p = 0$$

Together with the "linear plane wave" assumption

$$v^l = v_0^l + \varepsilon A^l e^{i(\omega t + k_j x^j)}$$

$$p = p_0 + \varepsilon p_1 e^{i(\omega t + k_j x^j)}$$

$$\rho = \rho_0 + \varepsilon \rho_1 e^{i(\omega t + k_j x^j)}$$

where the "background quantities" (index 0) are smooth.

Note: The physical fields follows from the real part of the complex solutions.

At linear order in ε we have the equations

$$i(\omega + k_j v_0^j)\rho_1 + i(k_j A^j)\rho_0 = 0$$

$$i(\omega + k_j v_0^j)A_l - \frac{i}{\rho_0} k_l p_1 = 0$$

To close the system, we need to provide an "equation of state". Let this define the speed of sound according to

$$\frac{p_1}{\rho_1} = c_s^2$$

Then we can eliminate the pressure and density variations to get

$$i(\omega + k_j v_0^j)A_i - \frac{i(k_j A^j)}{\omega + k_l v_0^l} c_s^2 k_i = 0$$

Contract this with k^i to get

$$\left[(\omega + k_j v_0^j)^2 - c_s^2 k^2\right](k_i A^i) = 0 \quad \Rightarrow \quad (\omega + k_j v_0^j)^2 - c_s^2 k^2 = 0$$

which is the dispersion relation for sound waves.

To interpret this result it is useful to return to the plane-wave Ansatz.

Considering the phase of the wave, we see that the phase-velocity follows from

$$\omega t + k_j x^j = \omega t + k\tilde{x} = \text{constant} \quad \Rightarrow \quad \sigma_p = \frac{d\tilde{x}}{dt} = -\frac{\omega}{k}$$

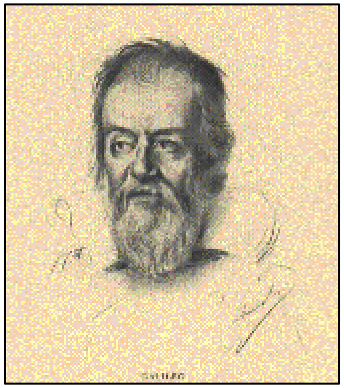
This means that we have

$$(\omega + k_j v_0^j)^2 - c_s^2 k^2 = 0 \quad \Rightarrow \quad \frac{\omega}{k} + \hat{k}_j v_0^j = \pm c_s \quad \Rightarrow \quad \sigma_p = \pm c_s + \hat{k}_j v_0^j$$

What does this tell us?

It shows that the waves travel at the speed of sound in a static medium, but that the wave speed is altered if there is a background flow.

It is, for example, possible for the flow to catch up with, and even overtake, the wave.



This, quite intuitive result, illustrates what we now call the <u>principle of relativity</u>.

It shows that velocities <u>add</u> in Newtonian physics.

This is an old idea, dating back to Galileo Galilei, making him the first relativist...

The speed of light

Now consider the analogous problem for Maxwell's equations. For simplicity, assume a non-conducting isotropic medium ($\sigma = \boldsymbol{j} = 0$). Then

$$D^i = \varepsilon E^i \quad \text{and} \quad H^i = B^i / \mu$$

and we have

$$\nabla_i E^i = 0$$

$$\nabla_i B^i = 0$$

$$\partial_t B^i + \varepsilon^{ijk} \nabla_j E_k = 0$$

$$\partial_t E^i - \frac{1}{\varepsilon\mu} \varepsilon^{ijk} \nabla_j B_k = 0$$

We can easily eliminate, for example, the electric field;

$$\partial_t^2 B^i + \varepsilon^{ijk} \nabla_j \partial_t E_k = \partial_t^2 B^i + \frac{1}{\varepsilon\mu} \varepsilon^{ijk} \nabla_j \left[\varepsilon_{klm} \nabla^l B^m \right]$$

$$= \partial_t^2 B^i + \frac{1}{\varepsilon\mu} \left[\underbrace{\nabla_j \nabla^i B^j}_{=0} - \nabla^2 B^i \right] = 0$$

We recognize the final equation

$$\partial_t^2 B^i - \frac{1}{\varepsilon\mu}\nabla^2 B^i = 0$$

as a wave-equation for the magnetic field. The wave speed is

$$c = \frac{1}{\sqrt{\varepsilon\mu}}$$

We (obviously) arrive at the same result by making the plane-wave Ansatz

$$B^i = A^i e^{i(\omega t + k_j x^j)} \quad \Rightarrow \quad \left[\omega^2 - \frac{k^2}{\varepsilon\mu}\right] A^i = 0 \quad \Rightarrow \quad \sigma_p = -\frac{\omega}{k} = \pm c$$

What do we learn from this?

First of all, electromagnetic disturbances propagate as waves.

The waves travel with the speed of light, c, but contrary to what we may have expected this speed appears to be independent of the observers velocity.

Maxwell's equations are not Galilean invariant…

Michelson-Morley experiment

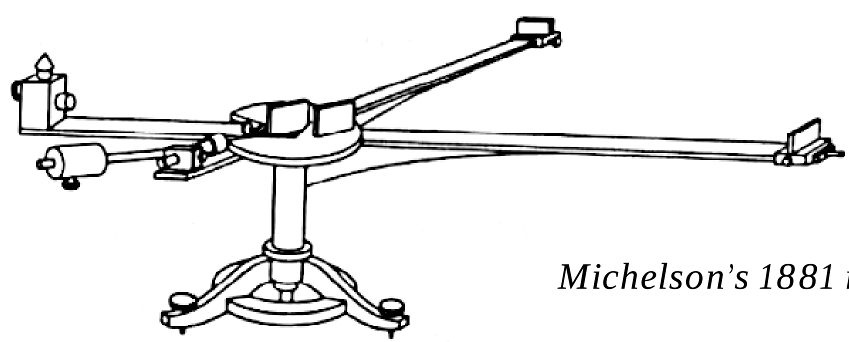
The notion that light corresponds to electromagnetic waves with a constant speed triggered a “crisis” in physics at the end of the 1800’s.

If light is a wave then it ought to be a vibration in something... This led to the idea of the all pervasive “aether”.

In order to test the idea, Michelson and Morley carried out a set of experiments designed to measure the motion of the Earth through the aether.

However, the experimental results were null.

Einstein interpreted this as evidence for a constant speed of light.



Michelson’s 1881 interferometer

Special relativity

Einstein's special theory of relativity describes how some basic measurable quantities - like time, distance, mass and energy – depend on the speed of the observer relative to the object being studied.

Guided by great physical insight, Einstein built the theory on two ideas:

– Principle of relativity: An inertial observer (in uniform linear motion) experiences the same physical laws as an observer at rest.

– The speed of light is a universal constant (fundamental law of physics).

If we take the speed of light to be constant we can convert time into distance, and vice versa.

We will often work in "geometric" units where c=1. Then we have

$$c = 1 = 3\times 10^{8}\ \mathrm{m/s} \quad \Rightarrow \quad 1\ \mathrm{s} = 3\times 10^{8}\ \mathrm{m}$$