Cosmological models

Brief recap

If the Universe is (spatially) homogeneous and isotropic then it is governed by Friedmann's equation for the scale factor *R(t)*;

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi\rho}{3} + \frac{\Lambda}{3} - \frac{k}{R^2} = \frac{8\pi\rho_{\mathrm{tot}}}{3} - \frac{k}{R^2}$$

where k is the constant curvature and Λ is the cosmological constant. We also have, for an equation of state $p=w\rho$,

$$\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{R}}{R}$$

and the three "typical" cases:

1) "matter" $w = 0 \quad \Rightarrow \quad \rho \propto R^{-3}$

2) "radiation", for which

$$p = \rho/3 \quad \Rightarrow \quad w = 1/3 \quad \Rightarrow \quad \rho \propto R^{-4}$$

3) "vacuum energy" (cosmological constant)

$$p = -\rho \quad \Rightarrow \quad w = -1 \quad \Rightarrow \quad \rho \propto R^0 = \text{constant}$$

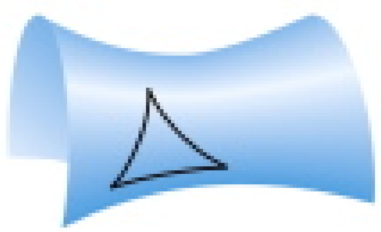
Curvature only

The simplest case is when the Universe is “empty” and there is no contribution to the energy density whatsoever. Then we simply have

$$\dot{R}^2 = -k$$

Clearly, flat space (*k=0*) is a solution. We can also have negative curvature. Such a Universe must be expanding or contracting with

$$\dot{R} = \pm 1 \quad \Rightarrow \quad R(t) = \pm R_0 \frac{t}{t_0}$$



(Subscript 0 indicates the present time.)

In such a Universe

$$t_0 = \frac{1}{H_0}$$

That is, the age of the Universe is exactly equal to the Hubble time.

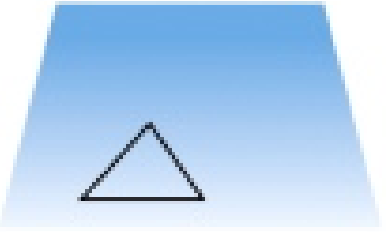
This model is known as the Milne Universe.

Spatially flat models

The empty Universe is, of course, not very relevant...

Let us instead consider flat models, for which *k=0*. Then we need to solve

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi\rho}{3} = \frac{8\pi\rho_0}{3} R^{-3(1+w)} \quad \Rightarrow \quad \dot{R}^2 = \frac{8\pi\rho_0}{3} R^{-(1+3w)}$$



Natural to try a power-law solution. This leads to

$$R(t) = R_0 \left(\frac{t}{t_0}\right)^{2/(3+3w)}$$

In this case, we find that the age of the Universe is related to the Hubble time via

$$t_0 = \frac{2}{3(1+w)} \frac{1}{H_0}$$

For $w > -1/3$ the Universe is younger than the Hubble time.

Matter/radiation only

For a matter only model $w=0$, and we have

$$R(t) = R_0 \left(\frac{t}{t_0} \right)^{2/3} \quad \text{and} \quad t_0 = \frac{2}{3} \frac{1}{H_0}$$

This is known as the Einstein-de Sitter model. Such a Universe would expand uniformly forever, although at a decreasing rate.

Eventually this model reaches the "Big Chill" (the temperature decreases with the expansion).

Until recently, this was the most favoured model for our Universe.

Meanwhile, for a pure radiation Universe we have $w = 1/3$ which means that

$$R(t) = R_0 \left(\frac{t}{t_0} \right)^{1/2} \quad \text{and} \quad t_0 = \frac{1}{2} \frac{1}{H_0}$$

Lambda only

Finally, let us consider the case when there is only a cosmological constant (the vacuum energy). Then we have w=-1, and the power-law solution is no longer valid.

Instead we need to solve

$$\dot{R}^2 = \frac{8\pi\rho_0}{3} R^2 \quad \Rightarrow \quad \dot{R} = H_0 R \quad \Rightarrow \quad R(t) = R_0 e^{H_0(t-t_0)}$$

Lesson: In a flat Universe, the cosmological constant can drive an exponential expansion.

It is easy to see that such a Universe would, in principle, be infinitely old.

Matter, curvature and Lambda

The models we have considered so far are all simplistic. We live in a multi-component Universe.

We need to consider the more complex problem where several terms contribute to the evolution of the scale factor.

Making the cosmological constant explicit (why?), we have

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi\rho}{3} + \frac{\Lambda}{3} - \frac{k}{R^2}$$

If we neglect radiation, then the energy density is entirely described by pressureless dust. In that case, we know that the total mass M of the Universe is constant. That is, we can introduce

$$C = 2M = \frac{8\pi\rho R^3}{3}$$

This allows us to write the Friedmann equation as

$$\dot{R}^2 = \frac{C}{R} + \frac{\Lambda}{3}R^2 - k \ , \quad C > 0 \ , \quad -\infty < \Lambda < \infty \ , \quad k = -1, 0, +1$$

Flat models (again)

This problem is not quite so easy to solve. To get an idea of the solutions one should expect, let us focus on the flat (*k=0*) case. Then we have

$$\dot{R}^2 = \frac{C}{R} + \frac{\Lambda}{3} R^2$$

There are 3 cases to consider, depending on the sign of Λ.

<u>Case 1.</u>

The simplest case is the Einstein-de Sitter model that we have already discussed. For Λ=0 we have

$$\dot{R}^2 = \frac{C}{R} \quad \Rightarrow \quad \frac{dR}{dt} = \frac{\sqrt{C}}{R^{1/2}} \quad \Rightarrow \quad R = \left(\frac{9Ct^2}{4} \right)^{1/3} = R_0 \left(\frac{t}{t_0} \right)^{2/3}$$

as before.

Case 2.

In the case of a positive cosmological constant, $\Lambda>0$, we have (note that this could represent a vacuum energy)

$$\dot{R}^2 = \frac{C}{R} + \frac{\Lambda}{3} R^2$$

Introducing a new dependent variable

$$u = \frac{2\Lambda}{3C} R^3 \quad \Rightarrow \quad \dot{u} = \frac{2\Lambda}{C} R^2 \dot{R}$$

we have

$$\dot{u}^2 = \frac{4\Lambda^2}{C^2} R^4 \dot{R}^2 = \frac{4\Lambda^2}{C^2} R^4 \left(\frac{C}{R} + \frac{1}{3} \Lambda R^2 \right) = \frac{4\Lambda^2 R^3}{C} + \frac{4\Lambda^3 R^6}{3C^2}$$

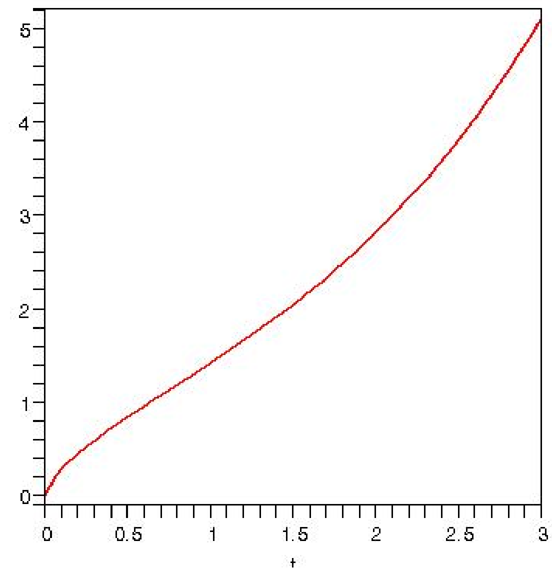
$$= 6\Lambda u + 3\Lambda u^2 = 3\Lambda \left(2u + u^2 \right)$$

Need to solve;

$$\frac{du}{dt} = \sqrt{3\Lambda} \left(2u + u^2 \right)^{1/2}$$

This is a separable equation…

$$\int_0^u \frac{du}{\left(2u+u^2\right)^{1/2}} = \sqrt{3\Lambda}\int_0^t dt = \sqrt{3\Lambda}\,t$$



Using

$$\int_0^u \frac{du}{\left[(u+1)^2-1\right]^{1/2}} = \int_0^v \frac{dv}{\left(v^2-1\right)^{1/2}} = \cosh^{-1} v$$

we have

$$u+1=\cosh\left(\sqrt{3\Lambda}\,t\right) \quad \Rightarrow \quad \frac{2\Lambda}{3C}R^3(t)=\cosh\left(\sqrt{3\Lambda}\,t\right)-1$$

and the final solution

$$R(t)=\left(\frac{3C}{2\Lambda}\right)^{1/3}\left[\cosh\left(\sqrt{3\Lambda}\,t\right)-1\right]^{1/3}$$

At late times, this Universe expands exponentially.

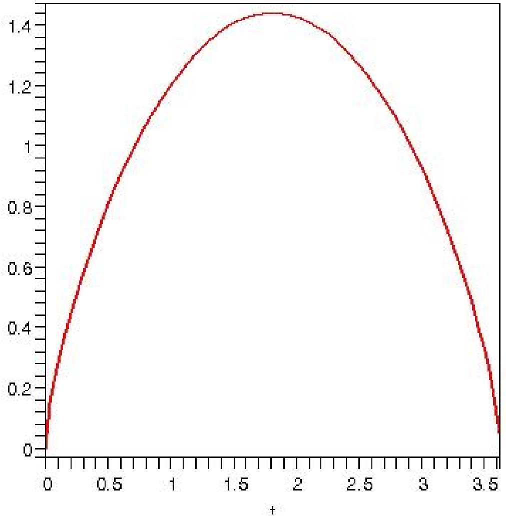
Case 3.

The problem for a negative cosmological constant, $\Lambda<0$, is similar (note that this would not be representing a vacuum energy).

Now using

$$u = -\frac{2\Lambda}{3C} R^3 \quad \Rightarrow$$

$$R(t) = \left(-\frac{3C}{2\Lambda}\right)^{1/3} \left[1 - \cos\left(\sqrt{-3\Lambda}\, t\right)\right]^{1/3}$$



This Universe reaches a maximum size and then contracts, eventually reaching a "Big Crunch".

It shrinks to zero as

$$t \to \frac{2\pi}{\sqrt{-3\Lambda}}$$

which could be very large if Λ is small.

Early times

It is worth noting that the early stage expansion is similar in the three cases. We had

$$\dot{R}^2 = \frac{C}{R} + \frac{\Lambda}{3} R^2$$

When R is small, the first term dominates, so

$$\dot{R}^2 \approx \frac{C}{R} \quad \Rightarrow \quad R \approx \left(\frac{9Ct^2}{4} \right)^{1/3}$$

as in the Einstein-de Sitter case.

This shows that a flat Universe expands at $t^{2/3}$ at early times.

In fact, the behaviour of the different solutions that we have obtained is easily deduced from the right-hand side of the Friedmann equation.

Other models (with k nonzero) are easily obtained in a similar fashion. Such models tend to show similar behaviour...

Inflation

Before we conclude the discussion of different cosmological models, it is appropriate to consider the inflation scenario.

It is generally thought that our Universe went through a period of very rapid expansion, known as inflation, in the early stages.

To understand the reason for this, we need to explain the so-called flatness problem.

To do this, we rewrite the Friedmann equation

$$H^2 = \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi\rho}{3} + \frac{\Lambda}{3} - \frac{k}{R^2}$$

in terms of;

$$\Omega_m = \frac{\rho}{\rho_0} = \frac{8\pi\rho}{3H^2} \qquad \text{Matter}$$

$$\Omega_\Lambda = \frac{\Lambda}{3H^2} \qquad \text{Cosmological Constant}$$

$$1 - \Omega = 1 - \left(\Omega_m + \Omega_\Lambda\right) = -\frac{k}{R^2 H^2}$$

At the present time, the parameters are related according to

$$1-\Omega_0 = -\frac{k}{R_0^2 H_0^2}$$

Observations (see next lecture) indicate that Ω_0 is very close to 1.

Since this suggests that $k=0$, it is called the flatness problem.

Why is it a problem?

If you extrapolate back in time, using the solutions that we have obtained, you find that Ω would have been extremely close to unity in the early Universe.

For example, at the transition from radiation to matter dominated evolution we would have

$$|1-\Omega| \sim 10^{-4}$$

Why would the Universe be so finely tuned?

We have already hinted at a solution to the problem.

Suppose there was an era of rapid expansion, as in a pure Λ Universe. Then we have seen that the scale-factor increases exponentially.

During such an era one would have

$$|1-\Omega| \sim e^{-2Ht}$$

so the Universe would flatten rapidly.

The inflationary scenario is usually explained in terms of a scalar field (the inflaton) which is associated with a value of $w < -1/3$.