Cosmology

Cosmological principle

Current cosmological models are based on the idea that the Universe "looks the same everywhere".

This statement, sometimes called the Copernican principle, may seem crazy at first sight. However, it makes sense on a large "average" scale where local variations in density are smoothed out.

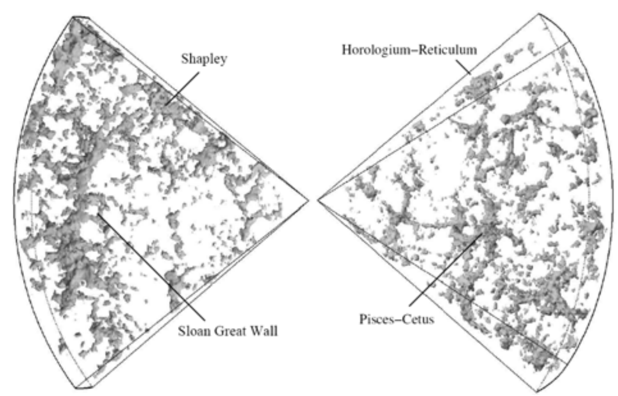
The idea is supported by the fact that the observed variations in the 3K cosmic microwave background are at the 10^{-5} level.

The Universe is said to be homogeneous and isotropic.

Homogeneous: There are no privileged points in spacetime

Isotropic: There are no privileged directions.

We will explore how we can use these general assumptions to build cosmological models.



Constant curvature

As a first attempt to understand the implications of the cosmological principle, let us consider a constant curvature spacetime.

Take as the starting point

$$R_{abcd} = K\left(g_{ad}g_{bc} - g_{ac}g_{bd}\right)$$

Then it follows that

$$R_{bd} = g^{ac}R_{abcd} = -3Kg_{bd} \quad \text{and} \quad R = g^{bd}R_{bd} = -12K$$

Use these results in the Einstein equations to get;

$$G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab} = 3Kg_{ab} = 8\pi T_{ab}$$

This places a contraint on the energy-momentum tensor.

Is this kind of model useful?

de Sitter

Before we address this key question, it is worth considering the three possible cases. Taking, first of all, $K=0$ we obviously have flat space.

For $K>0$, we have the so-called de Sitter model. The relevant line element can be written;

$$ds^2 = dt^2 - \alpha^2 \cosh^2\left(\frac{t}{\alpha}\right)\left[d\chi^2 + \sin^2\chi\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]$$

Written in this form, we see that the de Sitter model represents a Universe that shrinks to a mimimum size and then re-expands. Alternatively, one can use static coordinates. Then

$$ds^2 = \left(1 - Kr^2\right)dt^2 - \left(1 - Kr^2\right)^{-1} dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

This shows that there is a “cosmological horizon” when $Kr^2=1$.

The case $K<0$ leads to the anti-de Sitter model. We will not discuss it further, but this model has recently attracted quite a lot of attention from researchers trying to develop a quantum theory of gravity.

Vacuum energy vs Λ

In modern physics it is generally understood that vacuum does not really represent "empty space".

Suppose we introduce an isotropic vacuum energy ρ_{vac} such that

$$T^a{}_b = \rho_{\text{vac}}\delta^a_b \quad \Rightarrow \quad T_{ab} = \rho_{\text{vac}} g_{ab}$$

Compare to a perfect fluid;

$$T_{ab} = (p+\rho)u_a u_b - p g_{ab} \quad \Rightarrow \quad p_{\text{vac}} = -\rho_{\text{vac}}$$

In general, we can decompose the energy-momentum tensor into a matter part and a vacuum energy part. This leads to

$$G_{ab} = 8\pi\left(T_{ab}^{\text{matter}} + T_{ab}^{\text{vac}}\right) = 8\pi T_{ab}^{\text{matter}} + 8\pi\rho_{\text{vac}} g_{ab}$$

or

$$G_{ab} - 8\pi\rho_{\text{vac}} g_{ab} = G_{ab} - \Lambda g_{ab} = 8\pi T_{ab}^{\text{matter}}$$

Can represent the vacuum energy by a cosmological constant Λ (or vice versa).

Need to move on…

However…

The constant curvature models are not useful representations of the Universe.

They are simply not compatible with a dynamically interesting amount of matter and radiation.

This is further emphasized by the fact that the visible matter appears to be moving apart, meaning that the relative importance of the matter contribution was even greater in the past.

To make progress, we will relax the “perfect” cosmological principle and assume that the Universe is only spatially homogeneous and isotropic.

This leads us to the Robertson-Walker model.

World time

Use the same idea as before, but now assume that the model is only spatially homogeneous and isotropic.

This means that we need to introduce a suitable "time coordinate". To do this we adopt Weyl's postulate, which essentially says that we can model the matter in the Universe as a perfect fluid. Then we associate the time t with the geodesics of the "fluid elements".

This leads to the line element

$$ds^2 = dt^2 - h_{ij} dx^i dx^j \qquad i, j = 1-3$$

If we also assume that the 3-space is homogeneous and isotropic, then the time coordinate can only enter h_{ij} through a common factor. So we have

$$h_{ij} = \left[S(t)\right]^2 g_{ij}(x^i)$$

where $S(t)$ – which must be real – is called the scale factor.

Constant curvature (again)

It is easy to see that, if the 3-space is to be isotropic about every point, then it must be spherically symmetric about every point. This means that we can write (recall the derivation of the Schwarzschild solution)

$$g_{ij}dx^i dx^j = e^{\lambda(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

From this metric we find that the non-vanishing components to the Ricci tensor are

$$R_{11} = \lambda'/r \quad \text{and} \quad R_{22} = 1 + \tfrac{1}{2}re^{-\lambda}\lambda' - e^{-\lambda}$$

On the other hand, the constant curvature assumption now leads to (note sign!)

$$R_{ijkl} = K\left(g_{ik}g_{jl} - g_{il}g_{jk}\right) \quad \Rightarrow \quad R_{jl} = 2Kg_{jl} \quad \Rightarrow \quad \begin{cases} R_{11} = 2Ke^{\lambda} \\ R_{22} = 2Kr^2 \end{cases}$$

In other words, we have

$$\left.\begin{aligned} \lambda' &= 2Kre^{\lambda} \\ 1 + \tfrac{1}{2}re^{-\lambda}\lambda' - e^{-\lambda} &= 2Kr^2 \end{aligned}\right\} \quad \Rightarrow \quad e^{-\lambda} = 1 - Kr^2$$

Robertson-Walker

We have shown that the metric for a 3-space of constant curvature is

$$g_{ij}dx^i dx^j = \frac{dr^2}{1-Kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

where K can be positive, negative or zero. This means that the line element for relativistic cosmology can be written

$$ds^2 = g_{ab}dx^a dx^b = dt^2 - \left[S(t)\right]^2\left[\frac{dr^2}{1-Kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]$$

It is customary to rescale the radial coordinate and the scale factor in such a way that the curvature is parameterised by k=-1,0,+1.

This leads to the Robertson-Walker line element;

$$ds^2 = g_{ab}dx^a dx^b = dt^2 - \left[R(t)\right]^2\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]$$

Friedmann's equations

So far we have not considered the Einstein equations. To do this, we combine the Robertson-Walker spacetime with the postulated perfect fluid energy-momentum tensor.

We have

$$R_{ab} - \frac{1}{2} R g_{ab} - \Lambda g_{ab} = 8\pi T_{ab}$$

$$T_{ab} = \left(p + \rho \right) u_a u_b - p g_{ab}$$

where we choose "co-moving" coordinates, such that $u^a = (1, 0, 0, 0)$

The Einstein equations then lead to (here *R* is the scale factor!)

$$3\frac{\dot{R}^2 + k}{R^2} - \Lambda = 8\pi\rho$$

$$\frac{2R\ddot{R} + \dot{R}^2 + k}{R^2} - \Lambda = -8\pi p$$

We can rewrite these equations as

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi\rho}{3} + \frac{\Lambda}{3} - \frac{k}{R^2} = \frac{8\pi}{3}\left(\rho + \rho_{\text{vac}}\right) - \frac{k}{R^2}$$

$$\frac{\ddot{R}}{R} = -\frac{4\pi}{3}\left(\rho + 3p\right) + \frac{\Lambda}{3}$$

These are known as Friedmann's equations. First written down in 1922, they were not taken "seriously" until after Hubble's 1929 discovery that the Universe is expanding.

Einstein had introduced the cosmological constant earlier. He was looking for static solutions to the field equations. However, there was a problem...

A static solution demands that $\rho + 3p = 0$. This shows that, if the pressure vanishes then the energy density vanishes as well..

Also, if the energy density is positive then the pressure must be negative.

Hence, static solutions to the original Einstein equations did not appear to exist. To resolve the problem Einstein added the Λ term.

When the expansion of the Universe was discovered, he referred to this as his greatest blunder...

Combining these equations we find

$$\frac{d}{dt}\left(\rho R^3\right)+p\frac{d}{dt}\left(R^3\right)=0$$

If we now introduce the volume and the energy (recall $E=mc^2$) we get

$$\left.\begin{aligned} V &= \tfrac{4\pi}{3}R^3 \\ M &= \rho V \end{aligned}\right\} \quad \Rightarrow \quad \frac{dE}{dt}+p\frac{dV}{dt}=0$$

Comparing this to the 1st law of thermodynamics, we learn that the evolution of a Friedmann Universe is adiabatic (no heat generation).

Alternatively, rewrite the equation as

$$\dot{\rho}+3\left(p+\rho\right)\frac{\dot{R}}{R}=0$$

To make further progress, we need to provide an equation of state which relates p and ρ. In cosmology, it is generally assumed that the equation of state is linear.

This means that we have

$$p=w\rho$$

This leads to

$$\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{R}}{R} \quad \Rightarrow \quad \rho \propto R^{-3(1+w)}$$

Distinguish three cases:

1) “Matter”, usually taken as pressureless dust. Then

$$w = 0 \quad \Rightarrow \quad \rho \propto R^{-3}$$

and we see that the density decreases as the Universe expands. This is simply due to the fact that that total mass is constant.

2) “Radiation”. In this case we have

$$p = \frac{1}{3}\rho \quad \Rightarrow \quad w = \frac{1}{3} \quad \Rightarrow \quad \rho \propto R^{-4}$$

The energy density falls off faster than for matter because photons loose energy due to the redshift.

3) “Vacuum energy” (aka the cosmological constant)

$$p = -\rho \quad \Rightarrow \quad w = -1 \quad \Rightarrow \quad \rho \propto R^{0} = \text{constant}$$

At the present time, our Universe is <u>matter dominated</u>.

Cosmological parameters

The rate of expansion of the Universe

$$H = \frac{\dot{R}}{R}$$

is called the Hubble parameter. Its current value is

$$H_0 \approx 70 \pm 10 \text{ km/s/Mpc} \qquad \left[1 \text{ Mpc} \approx 3 \times 10^{24} \text{cm} \right]$$

It is also common to define the Hubble length

$$d_H = \frac{c}{H_0} \sim 10^3 \text{ Mpc}$$

This is a "typical cosmological scale". In fact, the Universe should be bigger than this.

Similarly, we can define the Hubble time (related to the "age" of the Universe)

$$t_H = \frac{1}{H_0} \sim 10^{10} \text{ yr}$$

The rate of change of expansion

$$q = -\frac{R\ddot{R}}{\dot{R}^2}$$

is known as the deceleration parameter. Recent measurements suggest that it is positive – the expansion of the Universe is accelerating.

Finally, it is common to introduce the density parameter

$$\Omega = \frac{8\pi\rho}{3H^2} = \frac{\rho}{\rho_{\text{crit}}}$$

Rewriting the Friedmann equation, we have

$$\left(\frac{\dot{R}}{R}\right)^2 = H^2 = \frac{8\pi\rho}{3} - \frac{k}{R^2} \quad \Rightarrow \quad \Omega - 1 = \frac{k}{H^2 R^2}$$

This shows that the sign of k depends on Ω, so the density parameter determines the nature of the Universe:

$$\rho < \rho_{\text{crit}} \quad \Rightarrow \quad k < 0 \quad \Rightarrow \quad \text{open}$$

$$\rho = \rho_{\text{crit}} \quad \Rightarrow \quad k = 0 \quad \Rightarrow \quad \text{flat}$$

$$\rho > \rho_{\text{crit}} \quad \Rightarrow \quad k > 0 \quad \Rightarrow \quad \text{closed}$$

The current value is close to 1.

The redshift

As usual, we obtain the equations for a light ray from the null geodesics. Consider purely radial motion;

$$ds^2 = dt^2 - \left[R(t)\right]^2 \frac{dr^2}{1-kr^2} = 0 \quad \Rightarrow \quad \frac{dt}{R(t)} = \pm \frac{dr}{\left(1-kr^2\right)^{1/2}}$$

Assume that a light ray is emitted from a star located at $r=r_1$ at time $t=t_1$ and then observed (by us) at $r=0$ at time $t=t_0$. Then

$$\int_{t_1}^{t_0} \frac{dt}{R(t)} = -\int_{r_1}^{0} \frac{dr}{\left(1-kr^2\right)^{1/2}} = \int_{0}^{r_1} \frac{dr}{\left(1-kr^2\right)^{1/2}}$$

Now consider a second light ray, emitted a time Δt_1 later, and arriving with a delay Δt_0. Since r_1 does not change, we have

$$\int_{t_1+\Delta t_1}^{t_0+\Delta t_0} \frac{dt}{R(t)} = \int_{0}^{r_1} \frac{dr}{\left(1-kr^2\right)^{1/2}}$$

This leads to

$$\int_{t_1}^{t_0}\frac{dt}{R(t)}=\int_{t_1+\Delta t_1}^{t_0+\Delta t_0}\frac{dt}{R(t)}\approx\int_{t_1}^{t_0}\frac{dt}{R(t)}+\frac{\Delta t_0}{R(t_0)}-\frac{\Delta t_1}{R(t_1)}\quad\Rightarrow\quad\frac{\Delta t_0}{R(t_0)}=\frac{\Delta t_1}{R(t_1)}$$

From this we can deduce that the ratio of the observed frequencies is determined by the ratio of the scale factors;

$$\frac{\Delta t_0}{\Delta t_1}=\frac{R(t_0)}{R(t_1)}\quad\Rightarrow\quad\frac{\nu_1}{\nu_0}=\frac{R(t_0)}{R(t_1)}$$

For an expanding Universe the frequency of the observed light is redshifted.

Comparing to the classical Doppler formula we have

$$\frac{\nu_1}{\nu_0}=1+z=\frac{R(t_0)}{R(t_1)}\quad\Rightarrow\quad z=\frac{R(t_0)-R(t_1)}{R(t_1)}\approx\frac{\dot{R}(t_0)}{R(t_0)}d_L=H(t_0)d_L$$

Where we have used the luminosity distance d_L. We learn that the redshift is proportional to the distance.