Rotating black holes

MATH3006
Relativity, black holes and cosmology

The Kerr solution

One would expect real astrophysical black holes to be rotating.

The rotating black hole solution to Einstein's equations in vacuum was discovered by Roy Kerr in 1963.

In terms of Boyer-Lindquist coordinates the Kerr line element is

$$ds^2 = \frac{\Delta}{\rho^2}\left(dt - a\sin^2\theta d\phi\right)^2 - \frac{\sin^2\theta}{\rho^2}\left[\left(r^2+a^2\right)d\phi - a\,dt\right]^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2$$

where

$$\Delta = r^2 - 2Mr + a^2$$

and

$$\rho^2 = r^2 + a^2\cos^2\theta$$

The Kerr black hole is described by two parameters: The mass M and the angular momentum per unit mass $a=J/M$.

Non-rotating limit

It is useful to consider the non-rotating limit of the Kerr solution.

Let $a \to 0$ to get

$$\Delta = r^2 - 2Mr \quad \text{and} \quad \rho^2 = r^2$$

and the line element becomes

$$ds^2 = \frac{\Delta}{\rho^2}\left(dt - a\sin^2\theta d\phi\right)^2 - \frac{\sin^2\theta}{\rho^2}\left[\left(r^2 + a^2\right)d\phi - a\,dt\right]^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2$$

$$\underset{a\to 0}{\longrightarrow} \left(1 - \frac{2M}{r}\right)dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

That is, we arrive at the Schwarzschild solution (in the usual form).

This shows that the Boyer-Lindquist coordinates are the "natural" extension for rotating black holes.

Flat space?

Let us now consider the limit when the mass M vanishes. This is a little bit artificial, but it provides some insight into the structure of the Kerr spacetime. We get;

$$ds^2 = \frac{\Delta}{\rho^2}\left(dt - a\sin^2\theta d\phi\right)^2 - \frac{\sin^2\theta}{\rho^2}\left[\left(r^2+a^2\right)d\phi - adt\right]^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2$$

$$\underset{M\to 0}{\longrightarrow} \quad \frac{r^2+a^2}{\rho^2}\left(dt - a\sin^2\theta d\phi\right)^2 - \frac{\sin^2\theta}{\rho^2}\left[\left(r^2+a^2\right)d\phi - adt\right]^2 - \frac{\rho^2}{r^2+a^2}dr^2 - \rho^2 d\theta^2$$

$$= dt^2 - \frac{r^2+a^2\cos^2\theta}{r^2+a^2}dr^2 - \frac{r^2+a^2\cos^2\theta}{r^2+a^2}dr^2 - \left(r^2+a^2\cos^2\theta\right)d\theta^2 - \left(r^2+a^2\right)\sin^2\theta d\phi^2$$

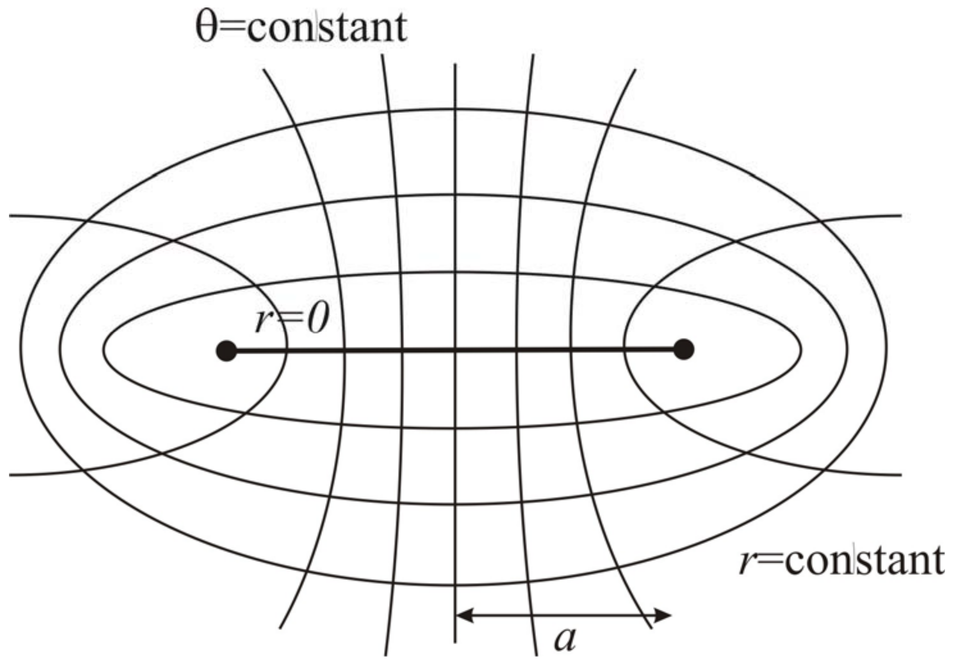
Expect this to be flat space… and it is, but in “funny” (ellipsoidal) coordinates.

We need

$$x = \left(r^2 + a^2\right)^{1/2} \sin\theta \cos\phi$$

$$y = \left(r^2 + a^2\right)^{1/2} \sin\theta \sin\phi$$

$$z = r\cos\theta$$



Properties

Now consider the general properties of this rotating solution. From the line element

$$ds^2 = \frac{\Delta}{\rho}\left(dt - a\sin^2\theta d\phi\right)^2 - \frac{\sin^2\theta}{\rho^2}\left[\left(r^2 + a^2\right)d\phi - a dt\right]^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2$$

we see that:

1. It is invariant under a rotation through ϕ. The solution is axially symmetric.

2. The metric coefficients do not depend on time t. The solution is stationary.

3. It is invariant under the transformations

   $t \to -t$ and $\phi \to -\phi$ or $t \to -t$ and $a \to -a$

   This suggests that the parameter a is associated with rotation.

4. The solution is asymptotically flat.

Two horizons

In Boyer-Lindquist coordinates the Kerr solution is singular when

$$\Delta = 0 \quad \text{and} \quad \rho^2 = 0$$

One can show that the curvature remains finite at $\Delta=0$. This means that this is a coordinate singularity. Meanwhile, the curvature diverges at $\rho^2=0$. This is a physical "curvature" singularity:

$$\rho^2 = r^2 + a^2 \cos^2\theta = 0$$

shows that this singularity is a ring of radius a.

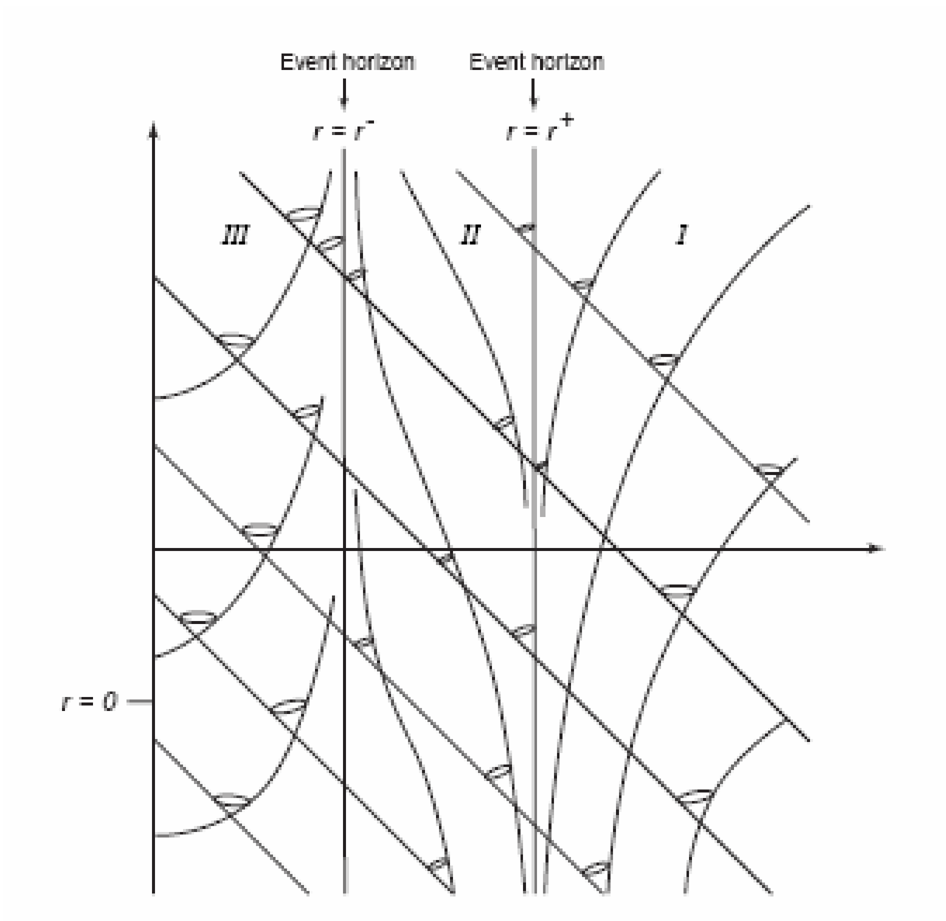
The event horizon follows from $\Delta=0$ (the radial coordinate becomes null). We find that the Kerr black hole has two horizons:

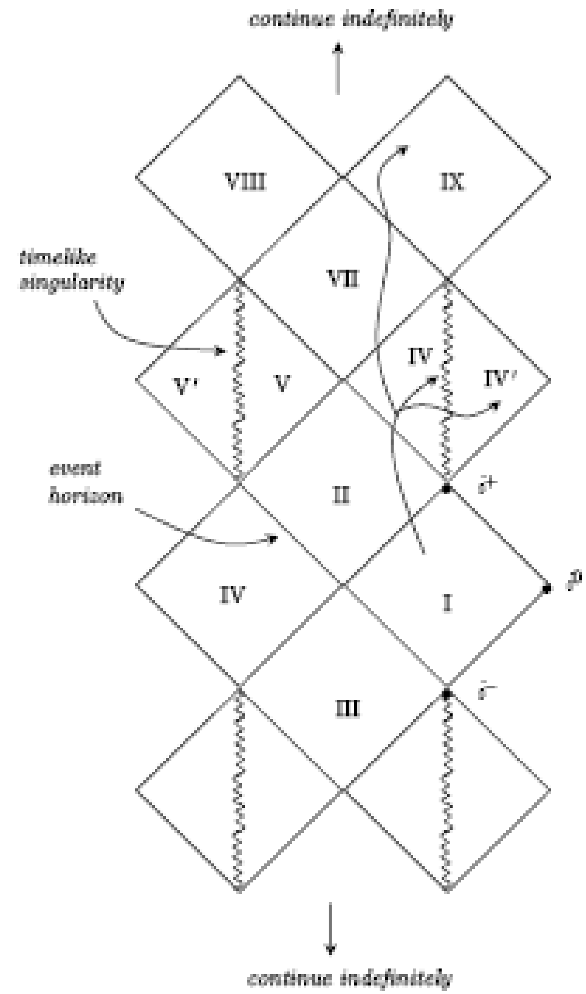
$$\Delta = r^2 - 2Mr + a^2 = 0 \quad \rightarrow \quad r_\pm = M \pm \left(M^2 - a^2\right)^{1/2}$$

We also have surfaces of infinite red-shift (coincided with the horizon in Schwarzschild) when

$$g_{tt} = 0 \quad \rightarrow \quad r^2 - 2Mr + a^2\cos^2\theta = 0 \quad \rightarrow \quad r_{s\pm} = M \pm \left(M^2 - a^2\cos^2\theta\right)^{1/2}$$

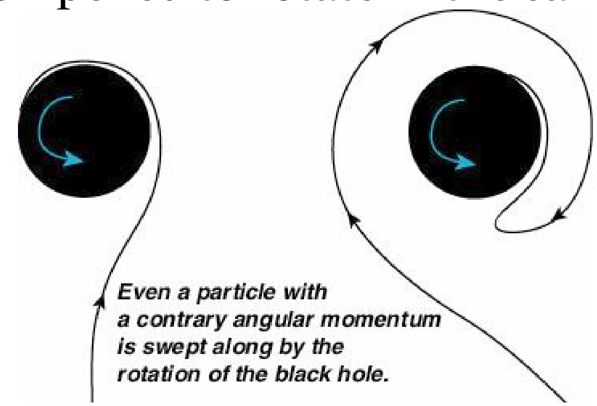
The causal structure of the Kerr spacetime is quite complicated…





Frame dragging

A Kerr black hole drags spacetime along with the rotation. This is called frame-dragging. Near the black hole this effect is very strong. In fact, in the region between r_+ and r_{s+} particles are compelled to rotate in the same direction as the black hole.



To see this, consider a photon emitted in the equatorial plane tangent to $r=r_{s+}$

$$dr = d\theta = 0 \quad \Rightarrow$$

$$ds^2 = g_{tt}dt^2 + 2g_{t\phi}dt\,d\phi + g_{\phi\phi}d\phi^2 = 0 \quad \Rightarrow \quad g_{tt} + 2g_{t\phi}\left(\frac{d\phi}{dt}\right) + g_{\phi\phi}\left(\frac{d\phi}{dt}\right)^2 = 0$$

$$r = r_{s+} \quad \Rightarrow \quad g_{tt} = 0 \quad \Rightarrow$$

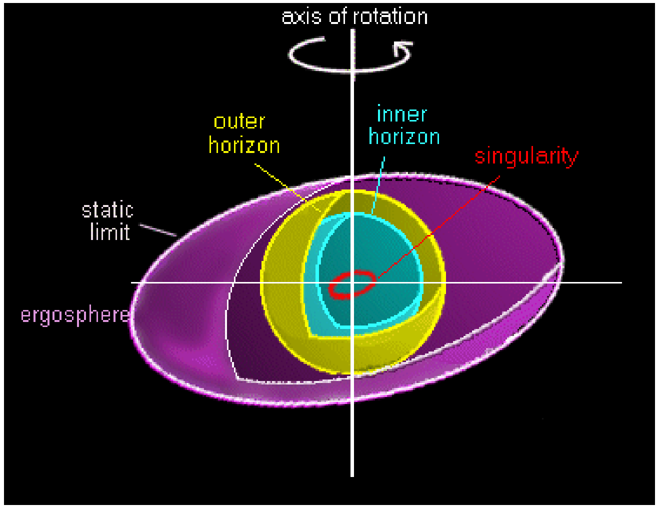
$$\frac{d\phi}{dt} = 0 \quad \text{or} \quad -\frac{2g_{t\phi}}{g_{\phi\phi}} > 0 \quad \text{for } a > 0 \quad \Rightarrow \quad \frac{d\phi}{dt} \geq 0$$

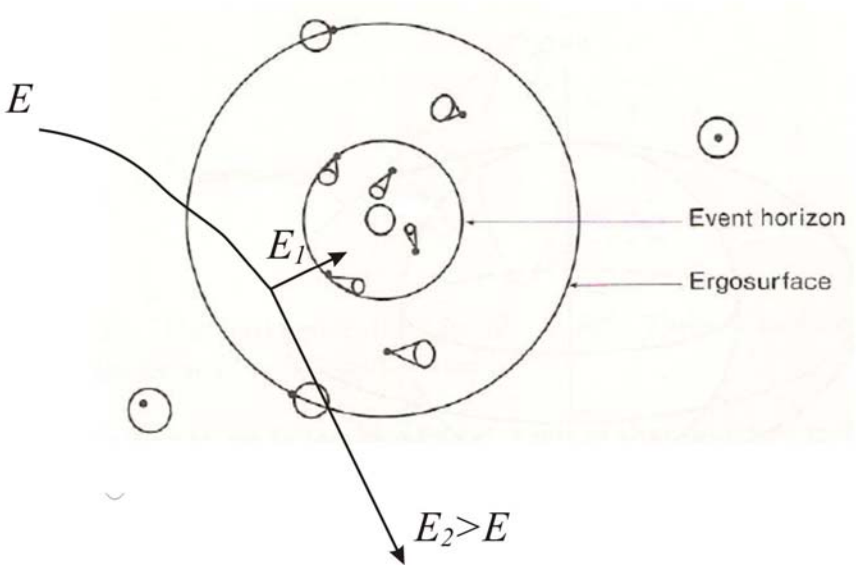
The photon can only move in one direction.

Ergosphere

This region of spacetime is known as the ergosphere.

$$r_{s+} = M + \left(M^2 - a^2 \cos^2 \theta\right)^{1/2} \geq r_+ = M + \left(M^2 - a^2\right)^{1/2}$$





Inside the ergosphere, particles can have "negative energy". In theory, this leads to a way to extract energy from the black hole. This is known as the Penrose process, and it would slow the black hole down.

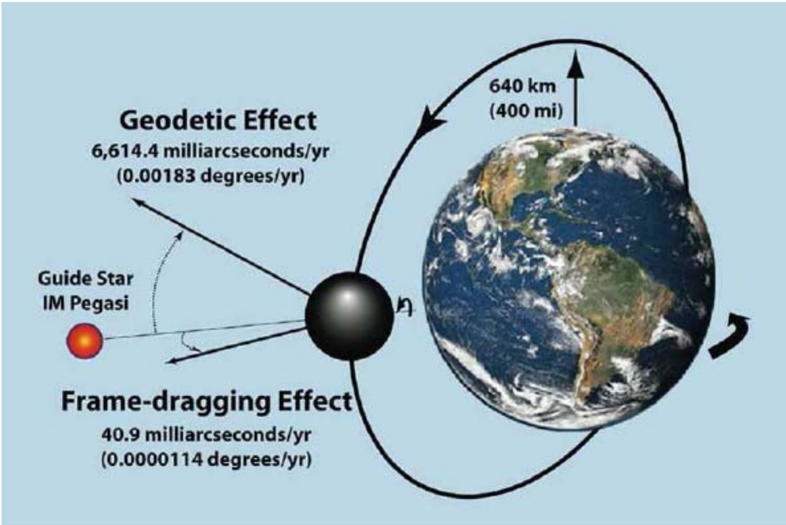
Gravity Probe B

The rotational frame-dragging has not yet been confirmed by experiment. This is essentially because the effect is very weak for objects in the solar system.

A project known as Gravity Probe B aims to measure the so-called Lense-Thirring precession for the Earth.

Gravity Probe B launched in April 2004 and will check, very precisely, tiny changes in the direction of spin of four gyroscopes in an Earth satellite orbiting at 600km altitude directly over the poles.

The latest data indicates clear evidence for the relativistic frame-dragging.



More ideas…

There are many other exciting ideas in black hole physics. Even though we don't have time to discuss them in any detail, it is worth mentioning some of them;

– A "classical" black hole has no hair; it can be described in terms of three parameters *M*, *a* and *Q*.

– Cosmic censorship: All singularities must be "clothed" by a horizon.

– Hawking radiation: A black hole is not really "black". Because of quantum pair creation near the horizon it emits radiation. This radiation can be described in terms of a temperature (proportional to $1/M$).

– Once we assign a temperature, we can relate it to an entropy. This has led to a set of laws of black-hole mechanics, analogous to the standard laws of thermodynamics.

– There is an ongoing debate concerning the information that is "lost" in a black hole. The entropy concept plays a key role in this discussion.