Null coordinates

Coordinates

We have seen that the Schwarzschild solution describes a non-rotating black hole. The event horizon, at $r=2M$, represents a one-way membrane through which no information can leak back into the exterior.

However, we also know that the standard Schwarzschild metric is singular at $r=2M$.

We will now discuss how this problem can be fixed by a clever choice of coordinates (we are dealing with a coordinate singularity not a physical singularity).

Before we do this, it is worth considering the nature of the coordinates.

Generally, one can show that the character of a given coordinate t, say, depends on

$$g^{tt} \begin{cases} > 0 & \text{timelike} \\ = 0 & \text{null} \\ < 0 & \text{spacelike} \end{cases}$$

For the Schwarzschild solution

$$g_{ab}dx^a dx^b = \left(1-\frac{2M}{r}\right)dt^2 - \left(1-\frac{2M}{r}\right)^{-1} dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

We see that:

– t is timelike for $r>2M$, but becomes spacelike inside the horizon,

– the radial coordinate r, which is spacelike for $r>2M$, becomes timelike inside the black hole.

This reflects the fact that, once inside the event horizon, all objects fall towards the centre.

It is also worth noting that r is a radial coordinate in the usual sense. The surface area of a 2-sphere is $4\pi r^2$.

Eddington-Finkelstein

To remove the singularity at $r=2M$, we change the time-coordinate:

$$t \to \bar{t} = t + 2M \ln(r - 2M)$$

Then

$$d\bar{t} = dt + \frac{2M}{r-2M} dr \quad \Rightarrow \quad dt = d\bar{t} - \frac{2M}{r}\left(1 - \frac{2M}{r}\right)^{-1} dr$$

We now have

$$dt^2 = d\bar{t}^2 - \frac{4M}{r}\left(1 - \frac{2M}{r}\right)^{-1} d\bar{t}dr + \frac{4M^2}{r^2}\left(1 - \frac{2M}{r}\right)^{-2} dr^2$$

This leads to

$$\left(1-\frac{2M}{r}\right)dt^2+\left(1-\frac{2M}{r}\right)^{-1}dr^2$$

$$=\left(1-\frac{2M}{r}\right)d\bar{t}^2-\frac{4M}{r}d\bar{t}dr+\left(1-\frac{2M}{r}\right)^{-1}\left(\frac{4M^2}{r^2}+1\right)dr^2$$

$$=\left(1-\frac{2M}{r}\right)d\bar{t}^2-\frac{4M}{r}d\bar{t}dr-\left(1+\frac{2M}{r}\right)dr^2$$

The Schwarzschild line element becomes

$$ds^2=\left(1-\frac{2M}{r}\right)d\bar{t}^2-\frac{4M}{r}d\bar{t}dr-\left(1+\frac{2M}{r}\right)dr^2-r^2\left(d\theta^2+\sin^2\theta d\phi^2\right)$$

which is regular at $r=2M$.

Note: The singularity at r=0 is a true physical singularity.

Our final result becomes simpler if we introduce the (ingoing) null coordinate

$$v = \overline{t} + r = t + r + 2M \ln\left(r - 2M\right) = t + r_*$$

where r_* is often called the "tortoise coordinate".

Then we get

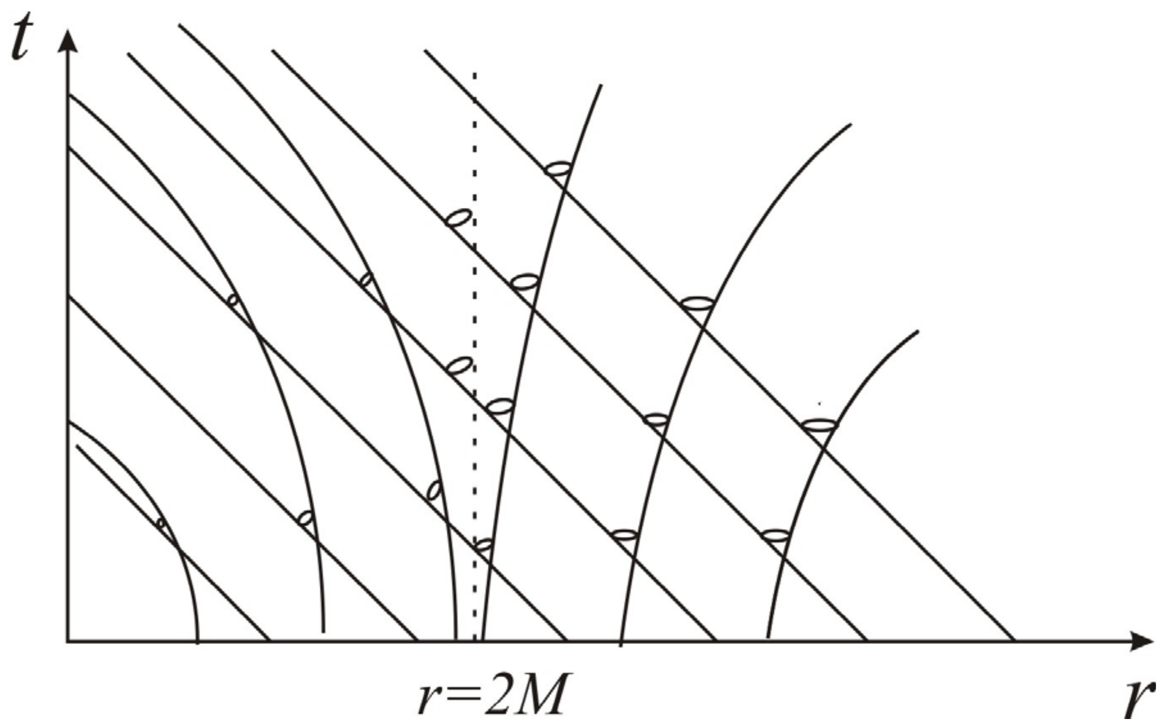
$$ds^2 = \left(1 - \frac{2M}{r}\right)dv^2 - 2dvdr - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

From this it is quite easy to show that $g^{vv} = 0$ so v is a null coordinate.

For radial null geodesics we get

$$\left(1 - \frac{2M}{r}\right)dv^2 - 2dvdr = 0 \quad \Rightarrow \quad \begin{cases} v = \text{constant} & \text{(ingoing)} \\ \dfrac{dv}{dr} = 2\left(1 - \dfrac{2M}{r}\right)^{-1} & \text{(outgoing)} \end{cases}$$

The ingoing Eddington-Finkelstein coordinate spacetime diagram becomes



Note: Even though the metric is regular at $r=2M$, the behaviour of the outgoing light rays is pathological at the horizon.

For outgoing light rays, we can define an analogous null coordinate;

$$u = t - r_*$$

in terms of which the line element becomes

$$ds^2 = \left(1 - \frac{2M}{r}\right) du^2 + 2dudr - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

In fact, we can combine the advanced coordinate v and the retarded coordinate u in such a way that the line element becomes

$$ds^2 = \left(1 - \frac{2M}{r}\right) dudv - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

Then r is a function which is implicitly given by

$$\frac{1}{2}(v - u) = r + 2M \ln(r - 2M)$$

This "double-null" form of the line element is very useful.

Kruskal-Szekeres

Focusing our attention on the case θ=constant, ϕ=constant, we have

$$ds^2 = \left(1 - \frac{2M}{r}\right) du dv$$

It is relatively easy to show that this is conformally flat. Introducing

$$t = \frac{1}{2}(v+u) \qquad x = \frac{1}{2}(v-u)$$

we have

$$ds^2 = \left(1 - \frac{2M}{r}\right) \underbrace{\left(dt^2 - dx^2\right)}_{\text{flat}}$$

The light cones are now all as in flat space, and both incoming and outgoing light rays are straight lines.

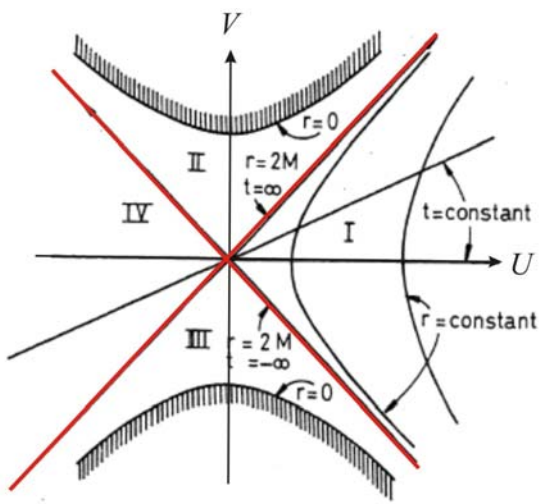
Making a further coordinate transformation, we arrive at the Kruskal-Szekeres form of the Schwarzschild solution;

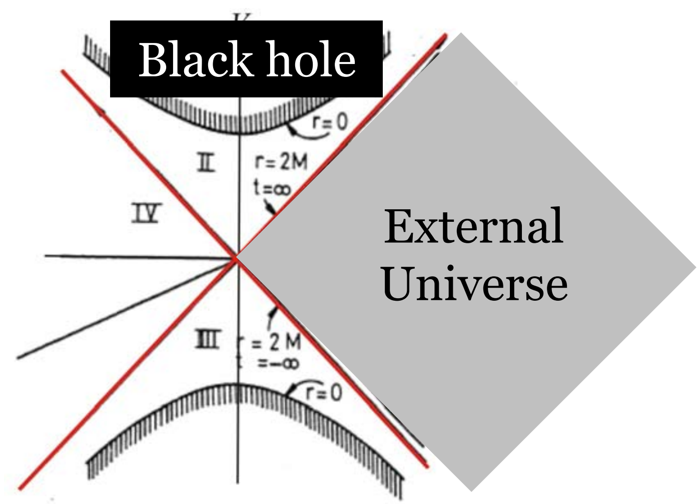
$$ds^2 = \frac{32M^3}{r}\exp\left(-\frac{r}{2M}\right)\left(dV^2 - dU^2\right) - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

where

$$V^2 - U^2 = \left(1 - \frac{2M}{r}\right)\exp\left(\frac{r}{2M}\right)$$

This represents the maximal analytic extension of the Schwarzschild metric.





Penrose-Carter

One great advantage of using null coordinates is that one can “compactify” the spacetime. The idea is to consider a new “unphysical” metric given by

$$\bar{g}_{ab} = \Omega^2 g_{ab}$$

and such that the points at infinity are brought to a finite position.

This leads to the so-called Penrose-Carter diagram.