Non-rotating black holes

Recap: Null geodesics

For light rays in the Schwarzschild spacetime we have

$$g_{ab}dx^a dx^b = \left(1-\frac{2M}{r}\right)dt^2 - \left(1-\frac{2M}{r}\right)^{-1} dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

Using

$$2L = g_{ab}\frac{dx^a}{ds}\frac{dx^b}{ds} = g_{ab}\dot{x}^a\dot{x}^b = 0$$

we have

$$2L = \left(1-\frac{2M}{r}\right)\dot{t}^2 - \left(1-\frac{2M}{r}\right)^{-1}\dot{r}^2 - r^2\left(\dot{\theta}^2 + \sin^2\theta\ \dot{\phi}^2\right) = 0$$

where s is a suitable affine parameter.

Radial light rays

Let us focus on <u>radial</u> light rays.

Then we have

$$\left(1-\frac{2M}{r}\right)\dot{t}^2-\left(1-\frac{2M}{r}\right)^{-1}\dot{r}^2=0$$

or

$$\left(\frac{dt}{dr}\right)^2=\left(1-\frac{2M}{r}\right)^{-2} \quad \Rightarrow \quad \frac{dt}{dr}=\pm\left(1-\frac{2M}{r}\right)^{-1}$$

Taking the plus sign and integrating, we find

$$t=r+2M\ln\left|r-2M\right|+\text{constant}$$

We see that *r=2M* divides the spacetime into two regions.

$$r>2M \quad \Rightarrow \quad r\to\infty \quad \text{as} \quad t\to\infty$$

$$r<2M \quad \Rightarrow \quad r\to 0 \quad \text{in a finite time}$$

With the other sign we find

$$t = -r - 2M \ln\left|r - 2M\right| + \text{constant}$$

Now we get

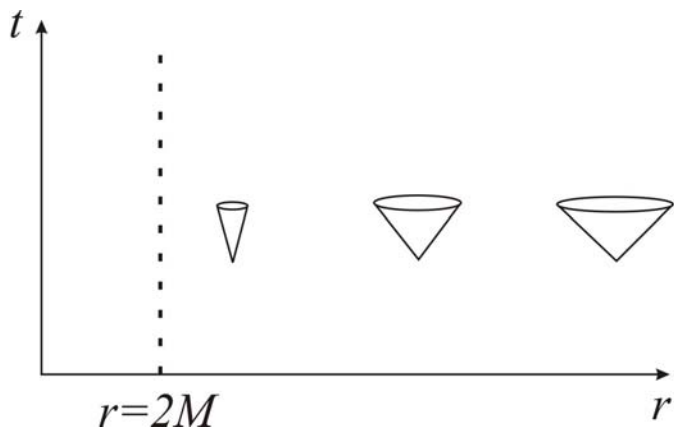
$$r > 2M \quad \Rightarrow \quad r \to 2M \qquad \text{as} \quad t \to \infty$$

$$r < 2M \quad \Rightarrow \quad r \to 2M \qquad \text{as} \quad t \to \infty$$

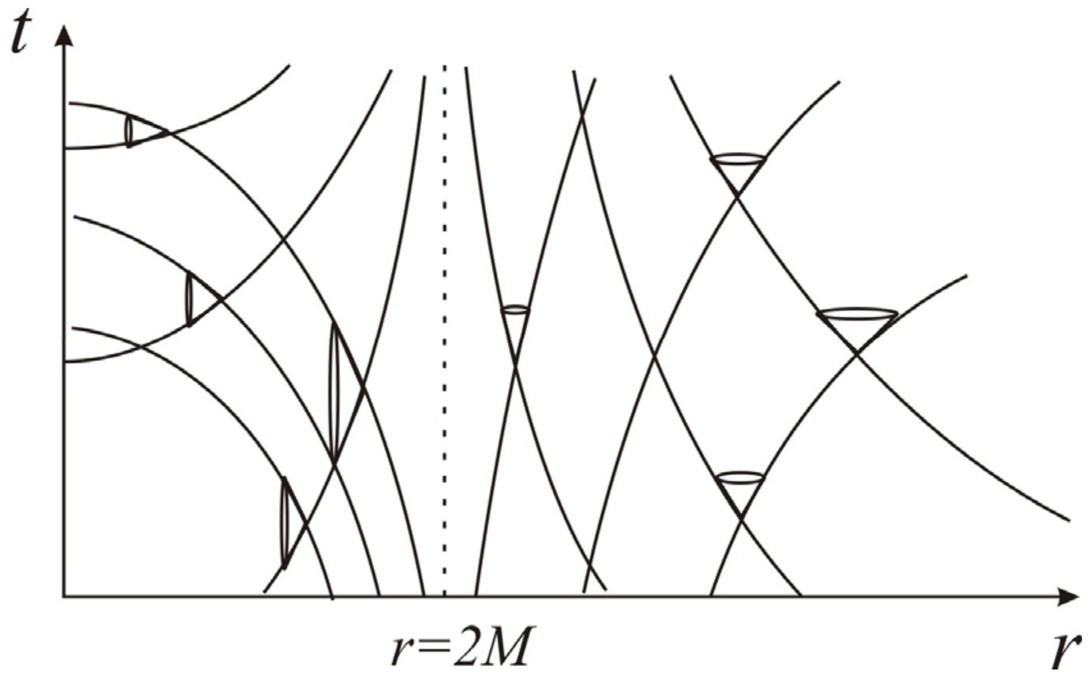
These results seem a bit peculiar, but can be understood if we consider the relevant spacetime diagram.

The light cone structure puts constraints on the possible history of an observer. An observer moves on a timelike world-line that must, at each point, lie within the future light cone.

As *r* approaches *2M*, the light cones close...



In fact, at *r=2M* the light cones "tilt over".



This suggests that, once inside *r=2M* light cannot escape to infinity.

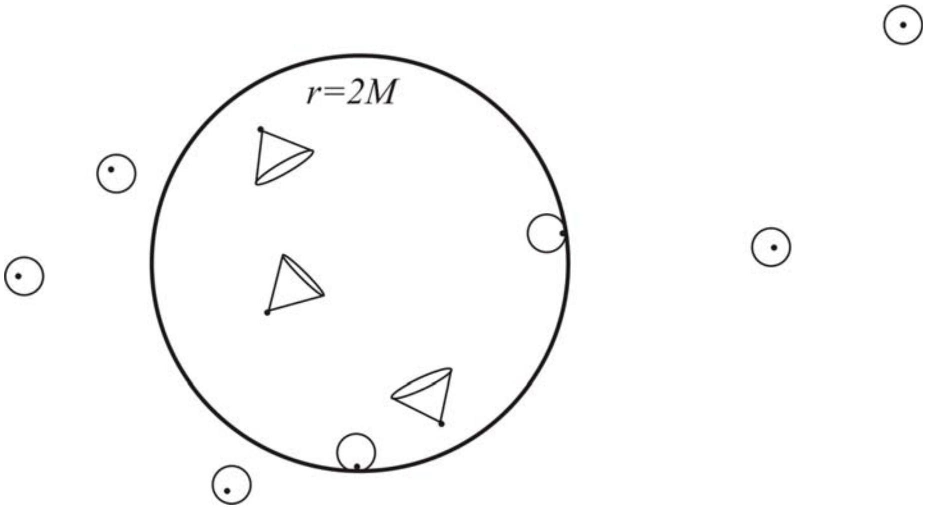
It is, however, not obvious that we can trust these results because the Schwarzschild metric is singular at *r=2M*;

$$g_{ab}dx^a dx^b = \left(1-\frac{2M}{r}\right)dt^2 - \boxed{\left(1-\frac{2M}{r}\right)^{-1}} dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

Black holes

The surface $r=2M$ is known as the event horizon. No signal emitted from inside $r=2M$ can reach the exterior.

This is a black hole.



A useful illustration is provided by spherical wavefronts in the equatorial plane. As we move closer to the origin the points from which the waves emanate are not longer at the centre of each wavefront.

All photons are dragged inwards, towards the singularity at the origin.

Infalling particles

We get a different perspective on this problem if we consider the radial motion of massive particles.

Test particles move on timelike geodesics, so we have

$$\frac{d}{d\tau}\left[\left(1-\frac{2M}{r}\right)\dot{t}\right]=0 \quad \Rightarrow \quad \left(1-\frac{2M}{r}\right)\dot{t}=E=\text{constant}$$

where τ is the proper time.

For radial motion we also have

$$\left(1-\frac{2M}{r}\right)\dot{t}^2-\left(1-\frac{2M}{r}\right)^{-1}\dot{r}^2=1$$

Combining the two results we have

$$E^2-\dot{r}^2=1-\frac{2M}{r}$$

If we suppose that the particle is released from rest at infinity, we must have

$$\dot{r} \to 0 \quad \text{as} \quad r \to \infty \quad \Rightarrow \quad E^2 = 1$$

Then the problem reduces to

$$\dot{r}^2 = \frac{2M}{r} \quad \Rightarrow \quad \frac{dr}{d\tau} = -\sqrt{\frac{2M}{r}} \quad \Rightarrow \quad \int d\tau = -\frac{1}{\sqrt{2M}} \int r^{1/2} dr$$

Note: sign is chosen such that the particle is infalling.

After integration, we obtain the radial position as a function of proper time;

$$\tau - \tau_0 = -\frac{1}{\sqrt{2M}} \frac{2}{3} \left(r^{3/2} - r_0^{3/2} \right)$$

This result (in fact, identical to the Newtonian result) shows that the particle reaches the origin in a <u>finite</u> proper time. Nothing special happens as the particle crosses $r=2M$.

Let us compare this result to what happens in <u>coordinate</u> time.

Then we need

$$\frac{dt}{dr} = \frac{\dot{t}}{\dot{r}} = -\left(\frac{r}{2M}\right)^{1/2}\left(1-\frac{2M}{r}\right)^{-1}$$

This is not very easy to integrate, but after some work one can show that, if r is close to $2M$ then

$$r-2M \approx \left(r_0 - 2M\right)e^{-(t-t_0)/2M}$$

from which it follows that

$$t \to \infty \quad \Rightarrow \quad r-2M \to 0$$

In other words, it takes an <u>infinite</u> amount of coordinate time (t) to reach $r=2M$.

This demonstrates that the coordinate t is not very useful near the event horizon of a black hole.

Newtonian argument

As a slight aside, it is worth noting that one can argue for the existence of black holes also in Newtonian gravity.

Consider an object with mass M and radius R.

We can work out the escape velocity from the kinetic energy and the gravitational potential energy for a moving object.

The total energy

$$E = \frac{1}{2} m v^2 - \frac{GM}{r}$$

is conserved. For an object that reaches infinity, we must have $E=0$ (why?).

Thus, the escape velocity is given by

$$v_{\text{esc}}^2 = \frac{2GM}{R}$$

For a given mass M we can make the escape velocity as large as we want by making the central object more compact.

However, we will reach the speed of light when

$$c^2 = \frac{2GM}{R} \quad \Rightarrow \quad R = \frac{2GM}{c^2}$$

The gravitational field is then so strong that not even light can escape.

This is a “black hole”.

This idea was first suggest by John Michell over 200 years ago. However, it was abandoned as the particle theory of light fell out of fashion.

It is interesting to note the similarities between the Newtonian black hole and its relativistic counterpart.

In both cases, the event horizon is given by

$$R = \frac{2GM}{c^2}$$

The interpretation is, however, very different.

In the Newtonian picture, light rays can be emitted from the surface. They simply do not reach an observer at infinity.

In relativity the horizon is a <u>one-way</u> membrane that does not allow information to leak to the outside Universe.