Gravitational waves

Weak fields

The tests of Einstein's theory that we have considered so far have been carried out in the weak-field setting of the solar system.

Let us focus on this limit, and write the metric as a small deviation from the Minkowski spacetime;

$$g_{ab} = \eta_{ab} + h_{ab} + O\left(h^2\right)$$

We will assume that *h* is small (in a suitable sense) and keep only linear terms in all calculations. It follows that

$$g^{ab} = \eta^{ab} - h^{ab} + O\left(h^2\right)$$

We want to use this metric in the Einstein equations

$$R_{ab} - \frac{1}{2} g_{ab} R = 8\pi G T_{ab}$$

Note: In vacuum we have

$$g^{ab}\left(R_{ab} - \frac{1}{2} g_{ab} R\right) = -R = 8\pi G T^a_{\ a} = 8\pi G T = 0 \quad \Rightarrow \quad R_{ab} = 0$$

Starting from the linearised Riemann tensor

$$R_{abcd} = \frac{1}{2}\left(\partial_c \partial_b h_{ad} + \partial_d \partial_a h_{bc} - \partial_d \partial_b h_{ac} - \partial_c \partial_a h_{bd}\right)$$

and carrying out the Ricci contraction (note that we raise and lower indices with the flat metric!), we get

$$R_{bd} = \frac{1}{2}\eta^{ac}\left(\partial_c \partial_b h_{ad} + \partial_d \partial_a h_{bc} - \partial_d \partial_b h_{ac} - \partial_c \partial_a h_{bd}\right) =$$

$$= \frac{1}{2}\left(\partial_c \partial_b h^c{}_d + \partial_d \partial^c h_{bc} - \partial_d \partial_b \underbrace{h^c{}_c}_{=h} - \underbrace{\partial_c \partial^c}_{=\Box} h_{bd}\right) =$$

$$= -\frac{1}{2}\left(\Box h_{bd} + \partial_d \partial_b h - \partial_c \partial_b h^c{}_d - \partial_d \partial^c h_{bc}\right)$$

As in the electromagnetic field problem, we can simplify things by adjusting the coordinates (choosing the gauge). Since we have 4 coordinates, we can impose 4 conditions on the metric.

Before we specify the gauge, we introduce a new variable

$$\bar{h}_{ab} = h_{ab} - \frac{1}{2}\eta_{ab} h$$

Noting that,

$$\bar{h} = \eta^{ab}\left(h_{ab} - \frac{1}{2}\eta_{ab} h\right) = h - 2h = -h$$

We see that this variable simply has the sign of the trace reversed. (At a deeper level, this variable is motivated by the form of the Einstein tensor.)

If we also adopt the <u>Lorentz gauge</u>

$$\partial^a \bar{h}_{ab} = 0 \quad \Rightarrow \quad \partial^a h_{ab} - \frac{1}{2}\eta_{ab}\partial^a h = 0$$

we find that

$$R_{ab} = -\frac{1}{2}\left(\Box\bar{h}_{ab} + \frac{1}{2}\eta_{ab}\Box h\right) \quad \Rightarrow \quad R = -\frac{1}{2}\left(\Box\bar{h} + 2\Box h\right) = -\frac{1}{2}\Box h$$

and the Einstein equations become

$$\boxed{\Box\bar{h}_{ab} = -16\pi G T_{ab}}$$

Newtonian limit

In order to make contact with Newtonian gravity, we need to note that, in the weak-field regime;

$$\left|T_{00}\right| \approx \rho \gg \left|T_{0i}\right| \gg \left|T_{ij}\right|$$

Hence, we focus on

$$\Box \bar{h}_{00} = -16\pi G T_{00} \approx -16\pi G \rho$$

We also find that

$$\Box \bar{h} = -\Box h = -16\pi G T \approx -16\pi G \rho$$

From these results it follows that

$$\Box \bar{h}_{00} = \Box h_{00} - \frac{1}{2}\eta_{00}\Box h \approx -16\pi G \rho \quad \Rightarrow \quad \Box h_{00} \approx -8\pi G \rho$$

For low velocities $v << c$, this simplifies further

$$\partial_t \sim v\partial_x \quad \Rightarrow \quad \partial_t \ll v\partial_x \quad \Rightarrow \quad \Box \approx -\nabla^2 \quad \Rightarrow \quad -\nabla^2 h_{00} \approx -8\pi G \rho$$

Comparing to the standard Poisson equation gives the final result;

$$\nabla^2 \Phi = 4\pi G \rho \quad \Rightarrow \quad h_{00} \approx 2\Phi \quad \Rightarrow \quad g_{00} \approx 1 + 2\Phi$$

Gravitational waves

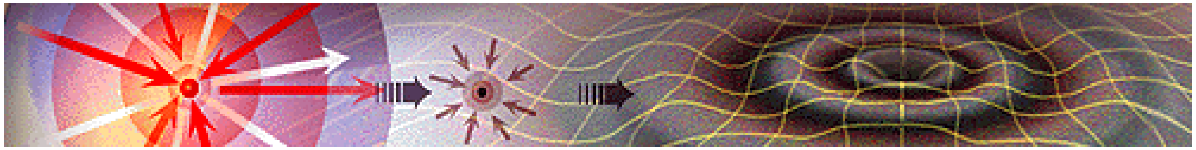
… but wait a second!

We have shown that, in the weak field limit, Einstein's equations reduce to a wave equation.

$$\Box \bar{h}_{ab} = \left(\partial_t^2 - \nabla^2\right)\bar{h}_{ab} = -16\pi G T_{ab}$$

In other words, General Relativity predicts that gravitational disturbances propagate as waves.

Spacetime is covered by tiny ripples, travelling at the speed of light.



Effect on matter

In order to work out what effect a gravitational wave has on the matter that it passes through, let us recall our discussion of geodesic deviation.

Measuring the distance between two geodesics (A and B), we had

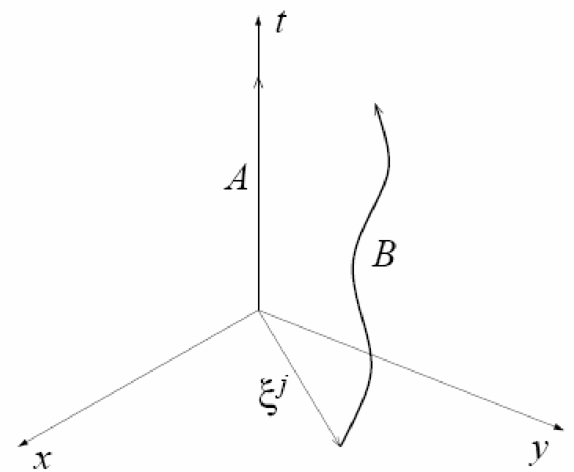
$$\frac{\partial^2 \xi^j}{\partial t^2} = -R^j{}_{0k0}\xi^k$$

Working in the local inertial frame of A, assume that B only deviates slightly from the original position. That is, take

$$\xi^j = x_0^j + \delta x^j$$

Then we get

$$\frac{\partial^2 \xi^j}{\partial t^2} = \frac{\partial^2 \delta x^j}{\partial t^2} \approx -R^j{}_{0k0} x_0^k$$



Now define

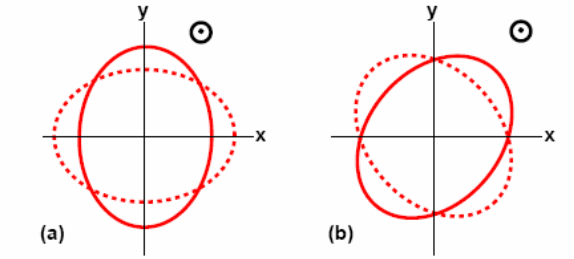
$$-\frac{1}{2}\frac{\partial^2 h_{ij}}{\partial t^2} = R_{i0j0}$$

to get

$$\frac{\partial^2 \delta x_j}{\partial t^2} \approx -\frac{1}{2}\frac{\partial^2 h_{jk}}{\partial t^2} x_0^k \quad \Rightarrow \quad \delta x_j \approx -\frac{1}{2} h_{jk} x_0^k$$

Taking the initial separation to be L and the maximum deviation to be ΔL, we see that

$$h = \frac{\Delta L}{L}$$



Gravity is tidal in nature, so gravitational waves change the distance between freely-moving bodies in empty space.

One can show that there are two wave polarisations.

The principles of detection

The typical displacement associated with astrophysical sources is

$$h = \frac{\Delta L}{L} \approx 10^{-21}$$

For L of order 100 km we need to resolve ΔL smaller than a proton diameter!

This makes detecting gravitational waves a real challenge.

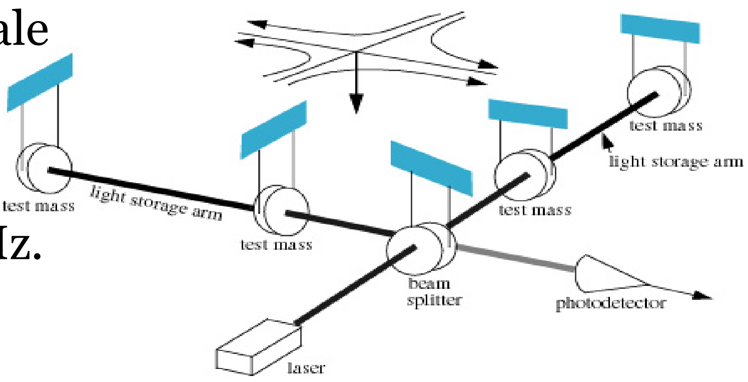
Over the last decade, a generation of large-scale laser interferometers have been developed.

– 1st generation are on-line now.

– Broadband sensitivity in the range 10-10^4 Hz.

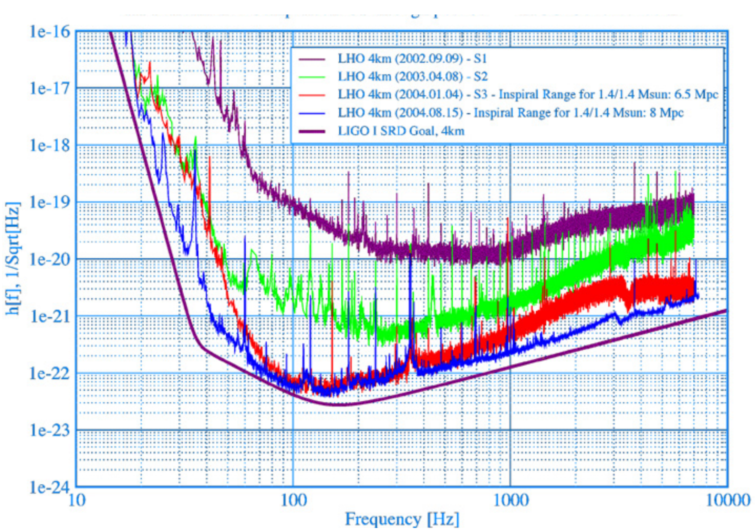
– Very long, 100 km folded into a few km.

– Upgrade to better technology in next few years.



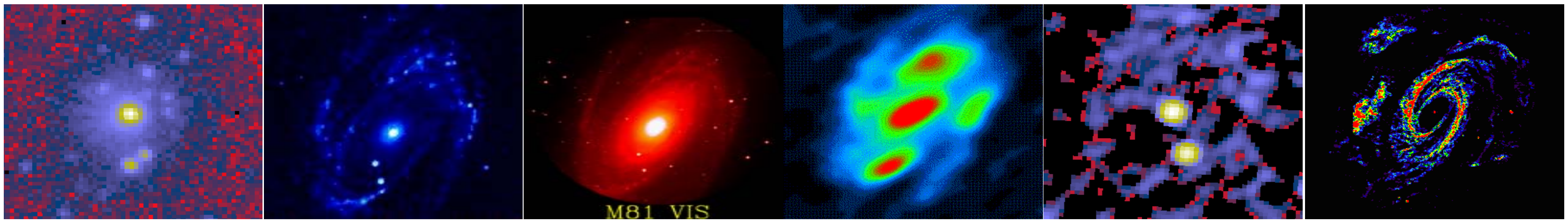
Ground based interferometers are limited by three fundamental noise sources:

– seismic noise at the lowest frequencies
– thermal noise at intermediate frequencies
– photon shot noise at high frequencies



The dark side of the Universe

There are many different ways to view the Universe (here the galaxy M81):



X-ray: 10 nm | UV: 200 nm | Optical: 600 nm | Infrared: 100 mm | Radio: 21cm | Radio – HI filter

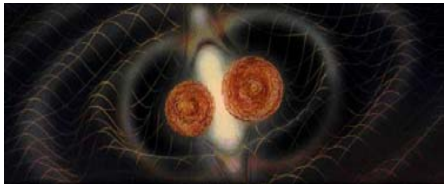
So far our information is based on electromagnetic waves at different wavelengths. Gravitational waves provide complementary information.

Electromagnetic waves	**Gravitational waves**
From individual particles	From bulk motion of matter
Scattered many times since generation	Couple weakly to matter, arrive in pristine condition
Imaging small fields of view	Detectors cover the entire sky
Wavelength smaller than source	Wavelength larger than source (no “imaging”)

Brief overview of sources

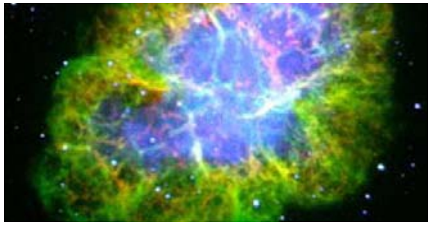
Compact binary inspiral: **"chirps"**

LIGO II: NS/NS inspiral at 300 Mpc
NS/BH at 650 Mpc
BH/BH at z=0.4



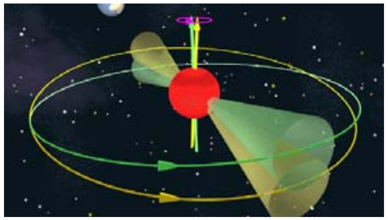
Supernovae / Gamma ray bursts: **"bursts"**

burst signals in coincidence with EM radiation
prompt alarm (~ one hour) with neutrino detectors
Black hole formation
convective boiling, neutron star pulsation



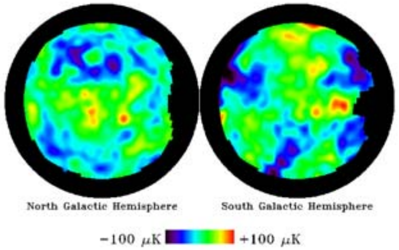
Pulsars in our galaxy: **"periodic"**

search for known pulsars
instabilities in newly born neutron stars
"lumpy" stars in accreting systems (LMXB)



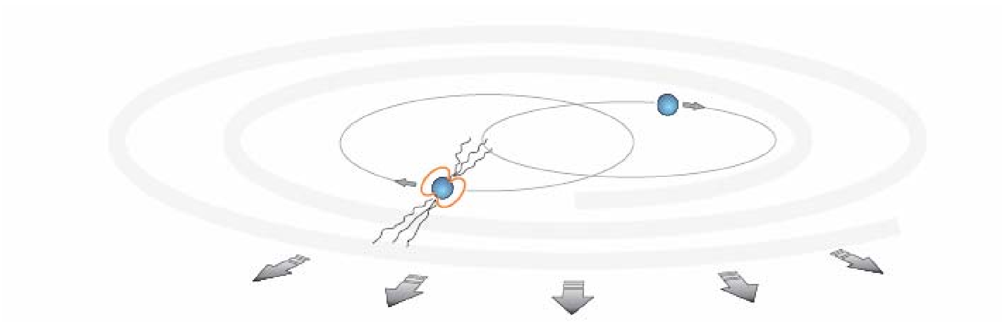
Cosmological signals: **"stochastic background"**

expect a stochastic GW background produced
when the Universe was younger than 10^{-24} seconds



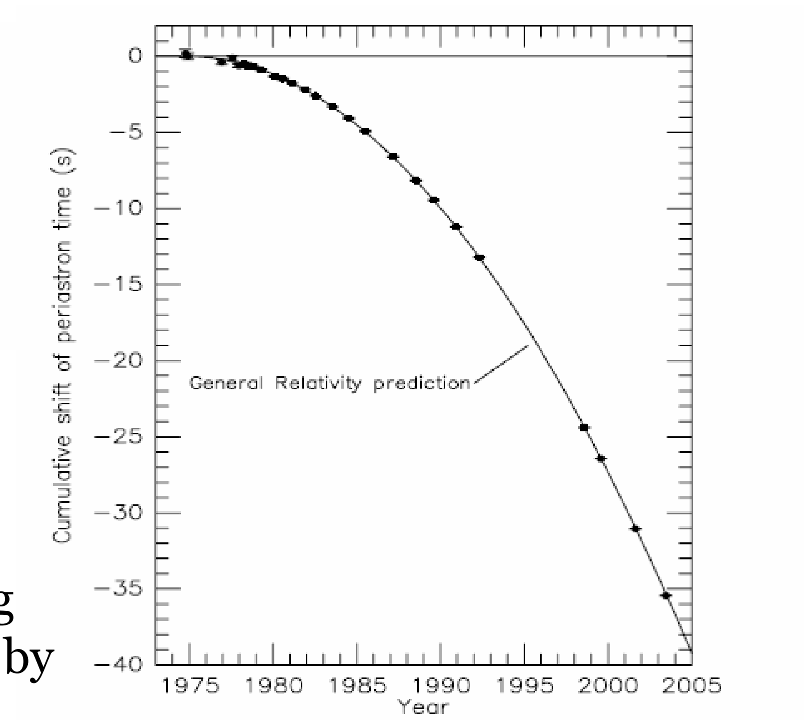
The binary pulsar

Pulsars are rotating neutron stars. As the star spins, the pulsar beam sweeps across the sky, and once every revolution it can be picked up by telescopes on Earth. Since their serendipitous discovery in 1967 around 2000 pulsars have been found.



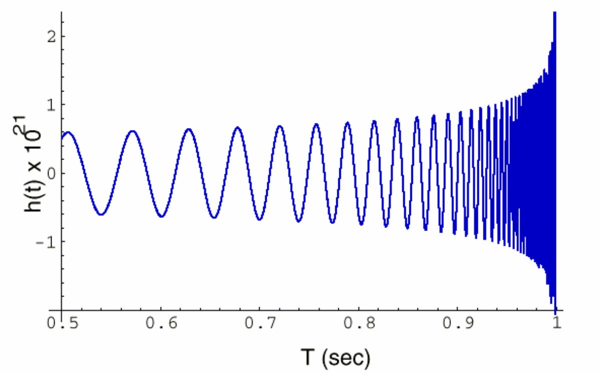
The so-called binary pulsar J1913+16 provides strong indirect evidence for the existence of gravitational waves.

After decades of monitoring, the shrinking of the orbit agrees with the rate predicted by Einstein's theory to better than 1%.

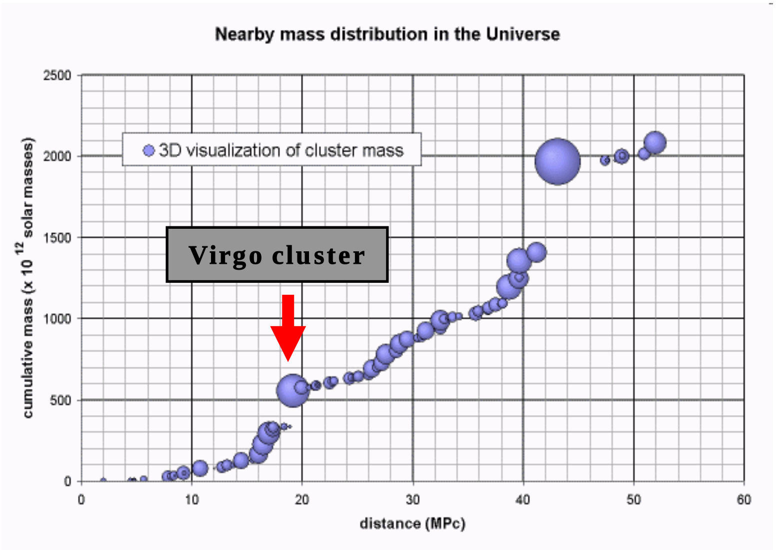


Inspiralling binaries

The waveform of a coalescing binary lasts 1000-10^4 cycles in the detectors sensitivity range. From the chirp signal one can infer: masses, spins, distance, and measure cosmological parameters



1st generation interferometers would be lucky to see 1/yr
2nd generation may see several hundred/yr



Advanced detectors:

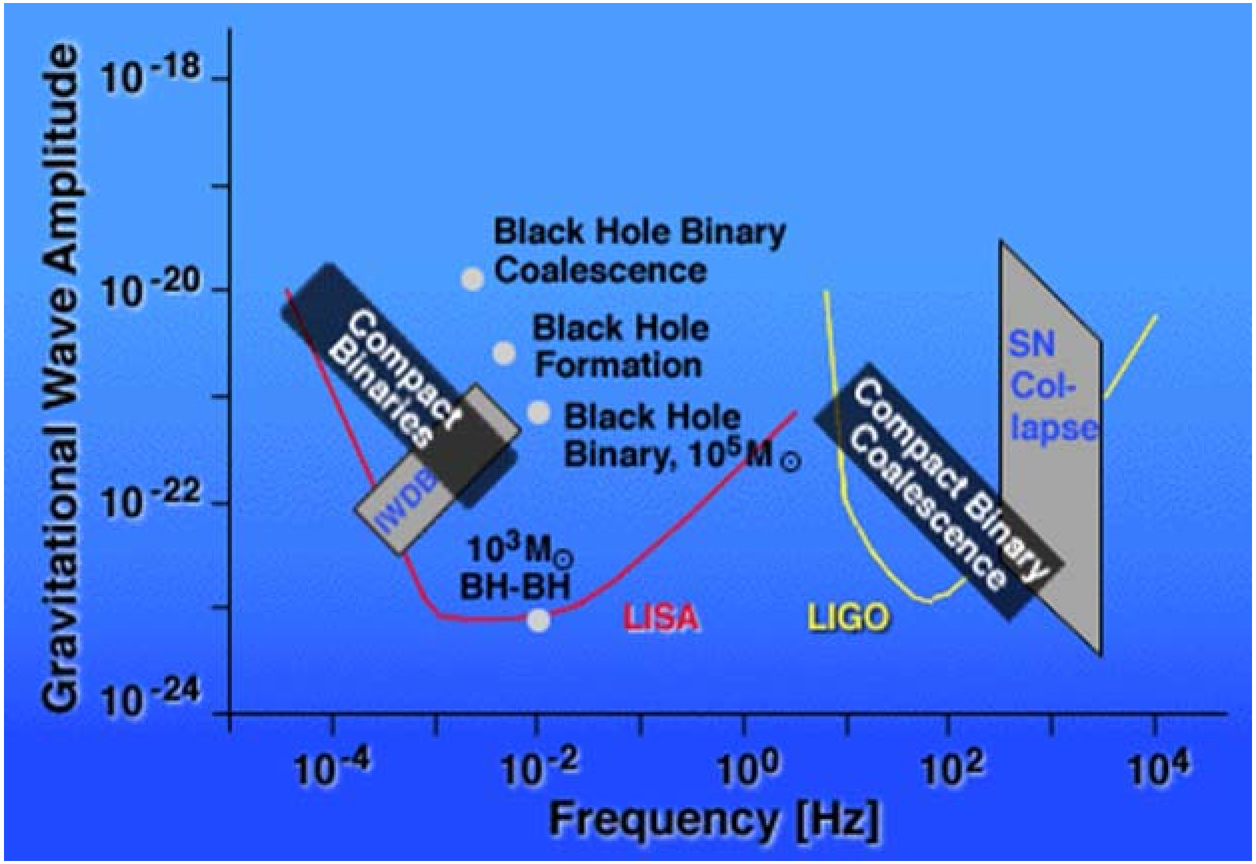
Improve sensitivity by a factor of 10x...

and the number of sources goes up 1000x!

Reaching the Virgo cluster of galaxies crucially enhances the event rate!

Gravitational-wave astronomy

Once the new window to the Universe is open we can hope to learn more about neutron stars, black holes, and the big bang.



Update on tests

Before we move on, it is relevant to discuss the current status of the various tests of General Relativity. Tests are usually described in terms the "deviation" from Einstein's theory.

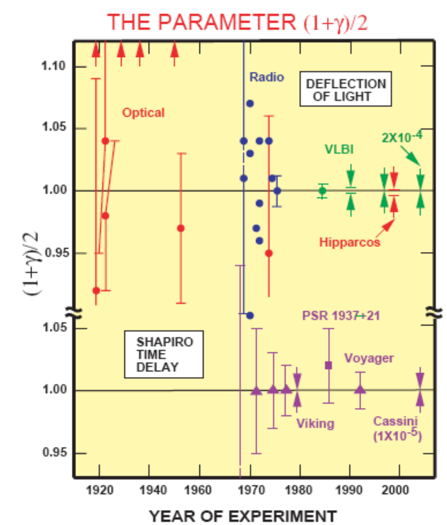
γ – how much curvature is produced by one unit of rest mass?
β – how much nonlinearity in superposition law for gravity?

The current "best" limits are:

$$|2\gamma - \beta - 1| \le 3\times 10^{-3}$$ Mercury perihelion shift

$$|\gamma - 1| \le 2.3\times 10^{-5}$$ time delay, Cassini tracking

$$|\gamma - 1| \le 4\times 10^{-4}$$ light deflection, VLBI



In addition, we have
– indirect evidence for the existence of gravitational waves
– strong evidence for the existence of black holes