Light bending, time delay and the gravitational red-shift

Null geodesics

Let us consider light rays in the Schwarzschild spacetime;

$$g_{ab}dx^a dx^b = \left(1-\frac{2M}{r}\right)dt^2 - \left(1-\frac{2M}{r}\right)^{-1} dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

This means that we consider <u>null</u> geodesics, such that

$$2L = g_{ab}\frac{dx^a}{ds}\frac{dx^b}{ds} = g_{ab}\dot{x}^a\dot{x}^b = 0$$

As before, this leads to

$$2L = g_{ab}\dot{x}^a\dot{x}^b = \left(1-\frac{2M}{r}\right)\dot{t}^2 - \left(1-\frac{2M}{r}\right)^{-1}\dot{r}^2 - r^2\left(\dot{\theta}^2 + \sin^2\theta\ \dot{\phi}^2\right)$$

The only difference is that we are no longer using the proper time as parameter. Instead, s is a suitable affine parameter.

Three of the Euler-Lagrange equations, remain as in the case of timelike geodesics.

These define the conserved angular momentum J and the energy E as

$$r^2\dot{\phi} = J = \text{constant} \qquad\qquad \left(1-\frac{2M}{r}\right)\dot{t} = E = \text{constant}$$

We also find that equatorial orbits, with q=p/2, remain in the equatorial plane.

The radial equation is, however, different. We now have

$$\left(1-\frac{2M}{r}\right)\dot{t}^2 - \left(1-\frac{2M}{r}\right)^{-1}\dot{r}^2 - r^2\left(\dot{\theta}^2 + \sin^2\theta\ \dot{\phi}^2\right) = 0$$

which leads to

$$\dot{r}^2 = E^2 - \left(1-\frac{2M}{r}\right)\frac{J^2}{r^2}$$

Taking a derivative of the final equation for equatorial null geodesics, we have

$$\ddot{r} - \frac{r-3M}{r^4} J^2 = 0$$

This shows that, for circular orbits with $r=R$=constant, we must have

$$r = 3M$$

This is known as the unstable photon orbit.

It is the only circular null geodesic that exists in the Schwarzschild geometry.

Light bending

As in the timelike case, we re-write the radial equation in terms of $u=1/r$ where $u=u(\phi)$. That is, we use

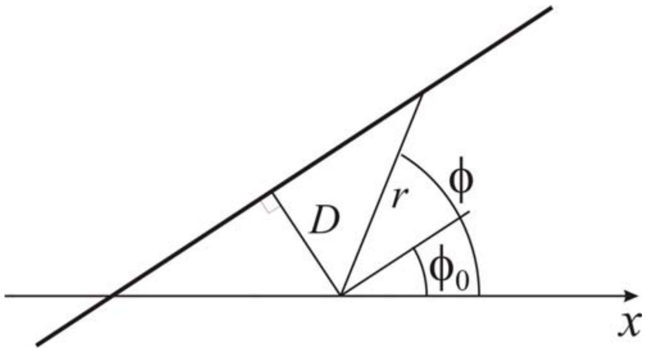
$$\dot{r} = \frac{dr}{du}\frac{du}{ds} = -\frac{1}{u^2}\frac{du}{ds} = -r^2\frac{du}{d\phi}\frac{d\phi}{ds} = -J\frac{du}{d\phi}$$

to get

$$\left(\frac{du}{d\phi}\right)^2 + (1-2Mu)u^2 = \frac{E^2}{J^2}$$

After taking another derivative we have

$$\frac{d^2u}{d\phi^2} + u = 3Mu^2$$



In flat space, $M=0$, we have the straight line solution;

$$u = \frac{1}{D}\sin(\phi - \phi_0)$$

To find an approximate solution to the curved spacetime equation, let us look for a solution of form (taking $\phi_0=0$ w.l.o.g.)

$$u = \frac{1}{D}\sin\phi + 3Mu_1 \qquad \text{where} \qquad 3M \text{ is small (in a suitable sense)}$$

Then we need to solve

$$3M\left(\frac{d^2u_1}{d\phi^2}+u_1\right) = \frac{3M}{D^2}\sin^2\phi + \underbrace{\frac{6M}{D}\sin\phi\, u_1 + 9M^2u_1^2}_{\text{ignore}}$$

Rewrite this as

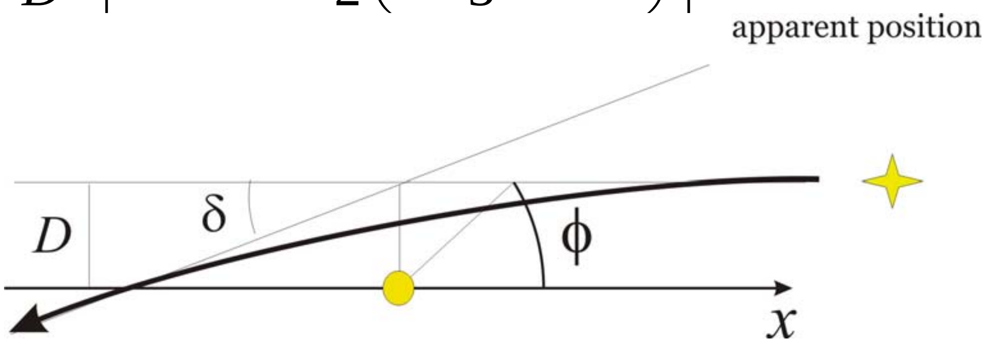
$$\frac{d^2u_1}{d\phi^2}+u_1 \approx \frac{1}{2D^2}\left(1-\cos 2\phi\right)$$

which has solution

$$u_1 = \frac{1}{D^2}\left[\underbrace{A\cos\phi + B\sin\phi}_{\text{homogeneous}} + \underbrace{\tfrac{1}{2}\left(1+\tfrac{1}{3}\cos 2\phi\right)}_{\text{particular integral}}\right]$$

The complete solution is;

$$u \approx \frac{1}{D}\left(1+\underbrace{\frac{3MB}{D}}_{\text{ignore}}\right)\sin\phi+\frac{3M}{D^2}\left[A\cos\phi+\frac{1}{2}\left(1+\frac{1}{3}\cos 2\phi\right)\right]$$



We are mainly interested in the accumulated effect, so consider the asymptotes. Since the incident ray is along the x-axis, we get

$$r \to \infty \quad \Rightarrow \quad u \to 0$$

$$\text{initially:} \quad \phi = 0 \quad \Rightarrow \quad \frac{3M}{2D^2}\left(1+\frac{1}{3}+2A\right)=0 \quad \Rightarrow \quad A=-\frac{2}{3}$$

$$\text{final:} \quad \phi = \pi+\delta \quad \Rightarrow \quad -\frac{\delta}{D}+\frac{3M}{2D^2}\left(1+\frac{1}{3}-2A\right)\approx 0 \quad \Rightarrow \quad \boxed{\delta=\frac{4M}{D}}$$

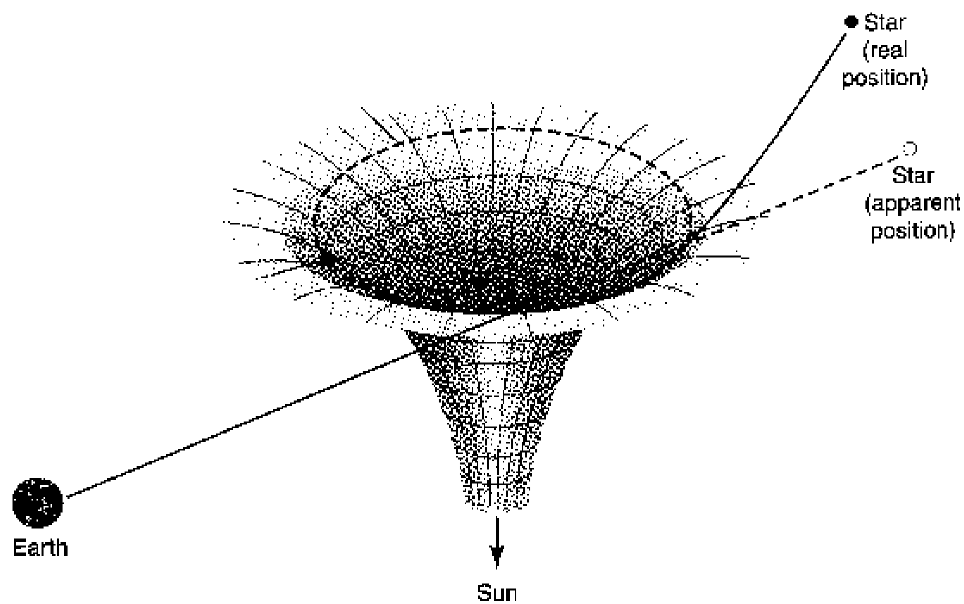
In normal units, the final deflection angle is

$$\delta \approx \frac{4GM}{Dc^2}$$

For light rays that graze the edge of the Sun, this works out to 1.75 arcseconds.

The effect was first confirmed by an expedition led by Eddington during the 1919 solar eclipse. The precision was, however, not very good.

The most accurate measurements of light bending have been carried out by long baseline radio interferometry for distant quasars.



Shapiro time delay

In 1964 Irwin Shapiro suggested a closely related effect, the so-called time delay. To see how this works, consider motion along a straight line in the equatorial plane (=leading order);

$$r\sin\phi = D \quad \Rightarrow \quad dr\sin\phi + r\cos\phi d\phi = 0 \quad \Rightarrow \quad d\phi = -\frac{D}{\sqrt{r^2 - D^2}}dr$$

Using this in the Schwarzschild line element we get

$$ds^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \left[\left(1 - \frac{2M}{r}\right)^{-1} + \frac{D^2}{r^2 - D^2}\right]dr^2 = 0$$

This leads to (expanding in *M/r*)

$$dt \approx \pm \frac{r}{\sqrt{r^2 - D^2}}\left[1 + \frac{2M}{r}\left(1 - \frac{D^2}{2r^2}\right)\right]dr$$

The last term in the brackets represents the time delay.

Lunar laser ranging has measured this effect with good precision.

Gravitational red-shift

Consider an observer at fixed radius R in the Schwarzschild spacetime, that emits a light signal. When emitted the signal has frequency ω_0. What is the frequency observed at infinity?

Generally, the frequency of a photon measured by an observer with four velocity u^a is

$$\omega = g_{ab} u^a \frac{dx^b}{ds}$$

In a sense, this relation defines the normalisation of parameter s.

For a static observer, the normalisation condition for the four velocity leads to

$$g_{ab} u^a u^b = g_{tt}(r)\left[u^t(r)\right]^2 = 1 \quad \Rightarrow \quad u^t(r) = \left(1 - \frac{2M}{r}\right)^{-1/2}$$

Combining these results we have

$$\omega = \left(1 - \frac{2M}{r}\right)^{1/2} \frac{dt}{ds} = \left(1 - \frac{2M}{r}\right)^{1/2} E$$

But the energy, E, is conserved along a geodesic. So we find that, for a photon emitted at r_1 and observed at r_2, the frequencies are related by

$$\frac{\omega_2}{\omega_1} = \left(\frac{1 - 2M/r_1}{1 - 2M/r_2}\right)^{1/2}$$

Taking $r_1 = R$ and $r_2 = \infty$ we find

$$\omega_\infty = \left(1 - \frac{2M}{R}\right)^{1/2} \omega$$

The observed frequency is lower - it has shifted towards the red.

The effect was first verified by Pound and Rebka in 1960, by firing gamma rays upwards a distance of 72 feet in the Earth's gravitational field.