The Schwarzschild solution

Symmetries

When Einstein first formulated the field equations of general relativity, he was convinced that they would be extremely difficult to solve.

In general, this is true. One would have to be able to handle nonlinear, coupled, partial differential equations with a large of terms.

However, many exact solutions to the equations are now known. In fact, solutions can be generated quite “easily” via computer algebra packages like Maple.

Unfortunately, most such solutions have no physical relevance.

Most useful solutions represent problems that are simplified because of symmetries.

The classic example is the Schwarzschild solution, which represents the exterior gravitational field of a non-rotating body.

This solution is static and spherically symmetric.

Stationary vs static

It is generally very difficult to find analytic solutions to the Einstein equations in situations representing systems with dynamics.

The problem is much simplified if one considers stationary or static systems.

Since we have the freedom to work with any set of coordinates we like, we need to introduce these familiar concepts carefully. We will take;

– stationary, to mean that the metric is time-independent (but evolutionary)

– static, to mean that there is no "motion" (time-reversal symmetric)

Mathematically, a spacetime is stationary if we there exists a coordinate system with time coordinate t such that

$$\frac{\partial g_{ab}}{\partial t} \doteq 0$$

If the spacetime is static, then it is also the case that $g_{0i} \doteq 0$.

Spherical symmetry

The form of the line element for a spherically symmetric spacetime can be deduced starting from the result for a 2-sphere with radius a;

$$ds^2 = a^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

The general line element should reduce to this form if we use “spherical” coordinates (t,r,θ,ϕ) and set t=constant, r=constant.

This leads to the general form

$$ds^2 = Adt^2 - 2Bdtdr - Cdr^2 - D\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

where the coefficients A, B, C and D are all functions of t and r.

Note: invariance under reflection means that

$$\theta \to \pi - \theta \quad \Leftrightarrow \quad d\theta = -d\theta$$

$$\phi \to -\phi \quad \Leftrightarrow \quad d\phi = -d\phi$$

which explains why there cannot be any cross terms of the form $dtd\theta$, $drd\phi$ etcetera.

Introducing new coordinates, we can rewrite the line element;

$$ds^2 = Adt^2 - 2Bdtdr - Cdr^2 - D\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

First, introduce a new radial coordinate

$$r' = \sqrt{D(t,r)} \quad \Rightarrow$$

$$ds^2 = A'dt^2 - 2B'dtdr' - C'dr'^2 - r'^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

Next, we would like to simplify the combination

$$A'dt^2 - 2B'dtdr' - C'dr'^2$$

say, by completing the square to get

$$d\tilde{t} = dt - \left(B'/A'\right)dr'$$

but... this is not a perfect differential. We need an integrating factor

$$I(t,r')d\tilde{t} = dt - \left(B'/A'\right)dr'$$

Then we get

$$A'dt^2 - 2B'dtdr' - C'dr'^2 = A'Id\tilde{t}^2 - A'I\left(\frac{C'}{A'} + \frac{B'^2}{A'^2}\right)dr'^2 = \tilde{A}d\tilde{t}^2 - \tilde{C}dr'^2$$

Defining

$$\tilde{A} = e^{\nu} \quad \text{and} \quad \tilde{C} = e^{\lambda}$$

and “dropping” all tildes and primes, we have the general spherically symmetric line element;

$$\boxed{ds^2 = e^{\nu} dt^2 - e^{\lambda} dr^2 - r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2 \right)}$$

where

$$\nu = \nu(t, r) \quad \text{and} \quad \lambda = \lambda(t, r)$$

Schwarzschild

Substituting the metric that leads to the general line element

$$ds^2 = g_{ab} x^a x^b = e^{\nu} dt^2 - e^{\lambda} dr^2 - r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2 \right)$$

into the vacuum Einstein equations we find the following three <u>independent</u> equations;

$$0 = G_0{}^0 = e^{-\lambda} \left(\frac{\lambda'}{r} - \frac{1}{r^2} \right) + \frac{1}{r^2} \qquad \text{(i)}$$

$$0 = G_0{}^1 = -e^{-\lambda} \frac{\dot{\lambda}}{r} \qquad \text{(ii)}$$

$$0 = G_1{}^1 = -e^{-\lambda} \left(\frac{\nu'}{r} + \frac{1}{r^2} \right) + \frac{1}{r^2} \qquad \text{(iii)}$$

We immediately see that

$$\lambda = \lambda(r)$$

So we have an ODE for λ;

$$e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} = 0 \quad \Rightarrow \quad \frac{d}{dr}\left(re^{-\lambda}\right) = 1 \quad \Rightarrow \quad re^{-\lambda} = r + \text{constant}$$

For reasons that will become clear later, we let the integration constant be *2m* to get

$$e^{-\lambda} = 1 - \frac{2m}{r} \quad \text{or} \quad e^{\lambda} = \left(1 - \frac{2m}{r}\right)^{-1}$$

Next, we subtract equations (i) and (iii) to get

$$\frac{e^{-\lambda}}{r}\left(\lambda' + \nu'\right) = 0 \quad \Rightarrow \quad \lambda' + \nu' = 0 \quad \Rightarrow \quad \lambda + \nu = h(t)$$

Hence, we have

$$e^{\nu} = e^{h(t)}\left(1 - \frac{2m}{r}\right)$$

and the line element becomes

$$ds^2 = \left(1 - \frac{2m}{r}\right)e^{h(t)}dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

Finally, introduce a new time coordinate such that

$$e^{h(t)}dt^2 = dt'^2 \quad \Rightarrow \quad t' = \int e^{h(t)/2}dt$$

to get the standard form for the Schwarzschild solution

$$\boxed{ds^2 = \left(1 - \frac{2m}{r}\right)dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)}$$

Weak field limit

In order to give an interpretation of *m* we need to consider the Newtonian, weak field, limit.

For weak gravitational fields and slow motion, one can show that

$$g_{00} \approx 1 + \frac{2\Phi}{c^2}$$

Using the gravitational potential

$$\Phi = -\frac{GM}{r}$$

we have

$$g_{00} \approx 1 - \frac{2GM}{rc^2} \quad \text{compared to} \quad g_{00} = 1 - \frac{2m}{r} \quad \Rightarrow \quad m = \frac{GM}{c^2}$$

Note: One usually works with "geometric" units where $G=c=1$, so $m=M$.

Final remarks

The Schwarzschild solution is;

– spherically symmetric

– static

– asymptotically flat (approaches Minkowski as $r \to \infty$)

Note: We did not impose the static condition. It came out of the calculation…

Birkhoff's theorem: A spherically symmetric and asymptotically flat solution is necessarily static.

Implication: No gravitational waves in spherical symmetry…