Maxwell’s equations
and
gravitational lensing

Maxwell's equations

In order to describe electromagnetic fields, we need a covariant form of Maxwell's equations.

Introduce an anti-symmetric tensor F^{ab} with components (in a local inertial frame)

$$F^{0i} = E^i \quad i = 1-3 \;\rightarrow\; (x, y, z)$$

$$F^{xy} = B^z \qquad F^{yz} = B^x \qquad F^{zx} = B^y$$

That is,

$$F^{ab} = \begin{pmatrix} 0 & E^x & E^y & E^z \\ -E^x & 0 & B^z & -B^y \\ -E^y & -B^z & 0 & B^x \\ -E^z & B^y & -B^x & 0 \end{pmatrix}$$

This is called the <u>electromagnetic field tensor</u>.

We also define the current density four vector;

$$j^a = \left(\rho, j^i\right) = \left(\text{charge density, charge current}\right)$$

With these definitions, Maxwell's equations can be written

$$\boxed{\begin{aligned} &\nabla_b F^{ab} = j^a \\ &\nabla_a F_{bc} + \nabla_c F_{ab} + \nabla_b F_{ca} = 0 \end{aligned}}$$

Note: We have "cheated" a little here... Strictly speaking we should have written the equations using partial derivatives (in the local inertial frame).

We are making use of the fact that if you know that a tensorial relation is valid in one coordinate system, then it is generally valid.

This allows us to simply replace the partial derivatives with covariant ones to get the curved spacetime form of the equations.

Sanity check:

Let us verify that these strange looking equations do, indeed, correspond to Maxwell's equations. Start by considering

$$\nabla_b F^{ab} = j^a$$

in a local inertial frame, i.e. a Cartesian coordinate system.

The time component leads to

$$\nabla_b F^{0b} = \nabla_i E^i = j^0 = \rho$$

Meanwhile, the three space components give us

$$\nabla_b F^{ib} = \partial_t F^{i0} + \nabla_j F^{ij} = j^i$$

For $i=1=x$, we have

$$\partial_t F^{x0} + \nabla_j F^{xj} = -\partial_t E^x + \left(\partial_y B^z - \partial_z B^y\right) = j^x$$

Recognize this as the x-component of

$$\nabla \times \boldsymbol{B} - \partial_t \boldsymbol{E} = \boldsymbol{j}$$

Another sanity check:

Next have a look at

$$\nabla_a F_{bc} + \nabla_c F_{ab} + \nabla_b F_{ca} = 0$$

This seems very complicated, but... due to the anti-symmetry of F^{ab} there are in fact only 4 equations.

Take $a=0$ to get

$$\partial_t F_{bc} + \nabla_c F_{0b} + \nabla_b F_{c0} = 0$$

See that b or c cannot be 0. Also... they must be different. Take $b=x$, $c=y$ to get

$$\partial_t F_{xy} + \partial_y F_{0x} + \partial_x F_{y0} = -\partial_t B^z - \partial_y E^x + \partial_x E^y = 0$$

Recognize this as the z-component of

$$\partial_t \boldsymbol{B} + \nabla \times \boldsymbol{E} = 0$$

Finally, take $a=x$ to get the final equation

$$\partial_x F_{yz} + \partial_y F_{zx} + \partial_z F_{xy} = \nabla_i B^i = 0$$

Energy-momentum tensor

In general relativity, all energy and stress affect the spacetime geometry. This means that an electromagnetic field will be coupled to the curvature.

To describe this, we need the energy-momentum tensor

$$T_{ab} = \frac{1}{4\pi}\left(-g^{cd}F_{ac}F_{bd} + \tfrac{1}{4}g_{ab}F_{cd}F^{cd}\right)$$

This is a complicated object, but some of the components may be familiar.

The energy density is

$$T_{00} = \frac{1}{8\pi}\left(\boldsymbol{E}^2 + \boldsymbol{B}^2\right)$$

We also have the momentum density;

$$T_{0i} \to -\frac{1}{4\pi}\boldsymbol{E}\times\boldsymbol{B}$$

which is known as the Poynting vector.

Vector potential

It is often useful to have a potential formulation for the electromagnetic field. We arrive at this by introducing a four vector A^a such that

$$F^{ab} = \nabla^b A^a - \nabla^a A^b$$

Using this we see that

$$j^a = \nabla_b F^{ab} = \nabla_b \left(\nabla^b A^a - \nabla^a A^b \right) =$$

$$= \nabla^2 A^a - \nabla^a \left(\nabla_b A^b \right) - g^{ac} \left(\nabla_b \nabla_c A^b - \nabla_c \nabla_b A^b \right)$$

$$= \nabla^2 A^a - \nabla^a \left(\nabla_b A^b \right) - g^{ac} R^b{}_{dbc} A^d = \nabla^2 A^a - \nabla^a \left(\nabla_b A^b \right) - R_c{}^a A^c$$

Working in vacuum ($j^a=0$) and Lorentz gauge, $\nabla_b A^b = 0$, we have the wave equation;

$$\Box A^a \equiv \nabla^2 A^a = -R^{ca} A_c$$

This shows that electromagnetic signals propagate as waves, and also gives us an idea of how the spacetime curvature affects the waves.

Wave propagation

Before we conclude the discussion about electromagnetism, let us take a brief look at the propagation of electromagnetic waves (=light) in vacuum.

We do this in the context of "geometric optics", i.e. we assume that the amplitude of the wave, A_{ab}, varies slowly compared to the phase S.

Then we can write

$$F_{ab} = A_{ab} e^{iS/\varepsilon} + O(\varepsilon)$$

To leading order in a small ε expansion we get

$$\nabla_b F^{ab} = 0 \quad \rightarrow \quad k_b A^{ab} = 0 \quad \rightarrow \quad \text{transverse wave}$$

where we have defined the wave vector $k_a = \nabla_a S$.

We also have

$$\nabla_a F_{bc} + \nabla_c F_{ab} + \nabla_b F_{ca} = 0 \quad \rightarrow \quad k_a A_{bc} + k_c A_{ab} + k_b A_{ca} = 0$$

Multiply this by k^a to get

$$\left(k^a k_a\right) A_{bc} = 0 \quad \rightarrow \quad k^a k_a = 0 \quad \text{wave vector is null}$$

Finally, note that

$$\nabla_a k_b = \nabla_a \nabla_b S = \nabla_a \partial_b S = \partial^2_{ab} S - \Gamma^c_{ba} S_c = \partial^2_{ba} S - \Gamma^c_{ab} S_c = \nabla_b \nabla_a S = \nabla_b k_a$$

Hence, it follows that

$$0 = \nabla_a \left(k_b k^b\right) = 2k^b \nabla_a k_b = 2k^b \nabla_b k_a$$

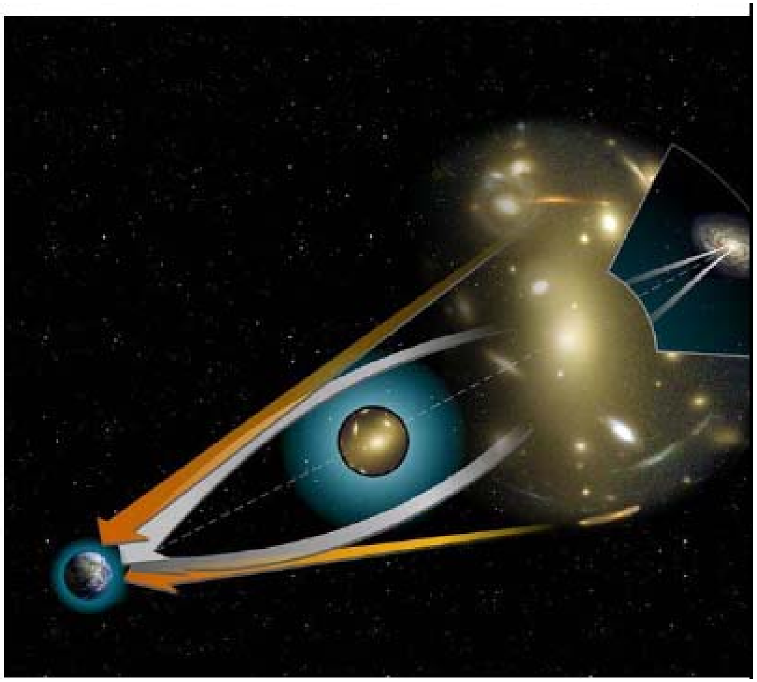
or

$$k^b \nabla_b k_a = 0$$

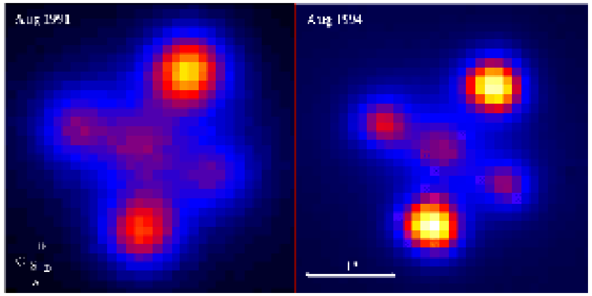
$\boxed{\text{Light moves along null geodesics.}}$

Gravitational lensing

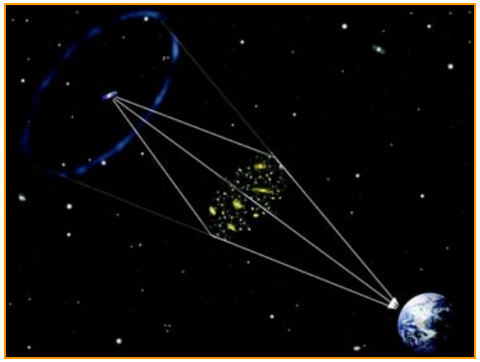
We now have an idea of how the spacetime curvature (i.e. gravity) affects light propagation. This leads to the notion of gravitational lensing.

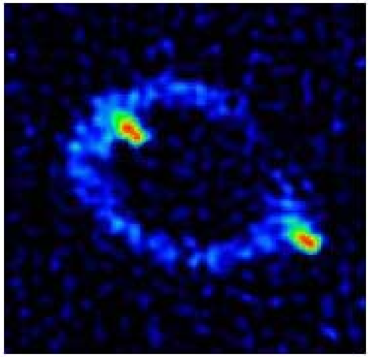


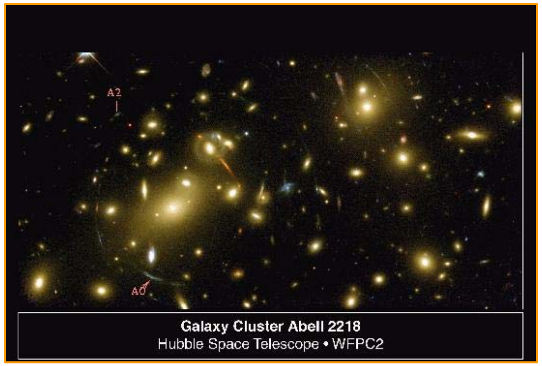
Can get multiple images...



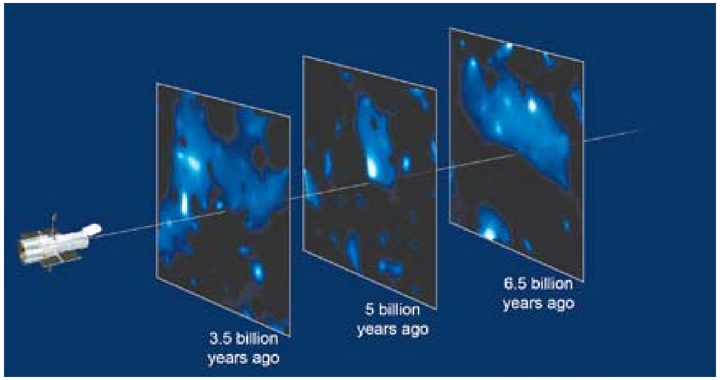
or rings (at least segments of)...

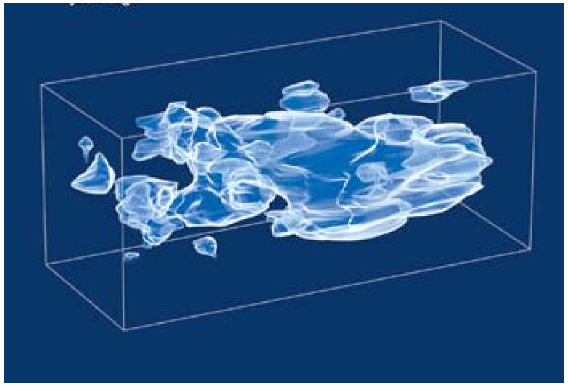






Lensing observations can be used to map the mass distribution in the Universe.





This is a powerful probe of dark matter.