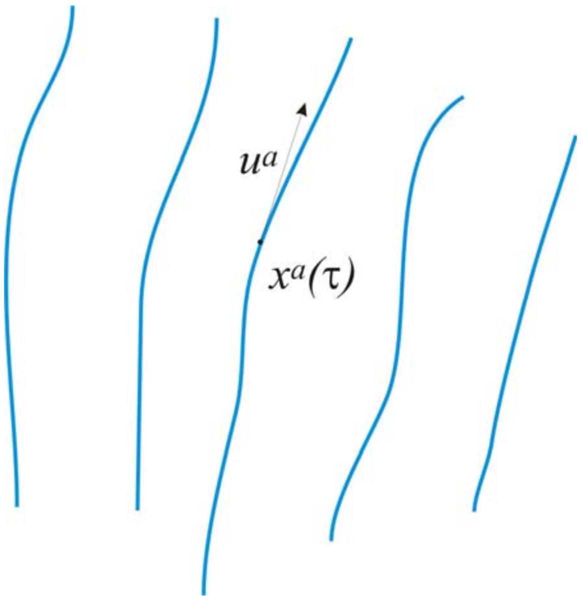
Fluids and dust

Dust

Consider the simplest example; a gas of non-interacting particles (density but no pressure). In relativity this is commonly called "dust"...

Let $x^a(\tau)$ be the world lines of the particles (parameterised by proper time);

$$u^a = \frac{dx^a}{d\tau}$$



If ρ_0 is the proper density of the dust (density measured by a co-moving observer) then the simplest 2nd rank tensor that we can form is

$$T^{ab} = \rho_0 u^a u^b$$

This is the energy-momentum tensor for dust.

What does this tell us?

First of all, we see that the energy density is [recall $u^a = \gamma(1, v^i)$]

$$T^{00} = \rho_0 \gamma^2$$

Interpret this;

$$m = \gamma m_0 \quad \text{and} \quad V = \frac{V_*}{\gamma} \quad \Rightarrow \quad \rho = \frac{m}{V} = \gamma^2 \frac{m_0}{V_*} = \gamma^2 \rho_0$$

see that T^{00} is the density measured by an observer at rest.

In general,

$$u^a u^b T_{ab}$$

is the energy density measured by an observer moving with four-velocity u^a.

Similarly, we have

$$T^{0b} = \rho_0 u^0 u^b = \gamma^2 \rho_0 (1, v^i) = \rho(1, v^i) = (\rho, \rho v^i)$$

or

$$T^{0b} = \left(\text{density, momentum density}\right)$$

which is the energy-momentum density measured by an observer at rest.

In general,

$$u^a T_a{}^b$$

is the momentum density measured by an observer moving with four-velocity u^a.

The dynamics is described by $\nabla_a T^{ab} = 0$.

This leads to

$$\nabla_a(\rho_0 u^a u^b) = u^b \nabla_a(\rho_0 u^a) + \rho_0 u^a \nabla_a u^b \qquad (\#)$$

Contract this with u_b to get

$$u_b\left[u^b \nabla_a(\rho_0 u^a) + \rho_0 u^a \nabla_a u^b\right] = \nabla_a(\rho_0 u^a) + \rho_0 u^a \underbrace{\left(u_b \nabla_a u^b\right)}_{=0} = 0$$

Now use this in (#) to see that

$$\rho_0 u^a \nabla_a u_b = 0$$

or

$$\frac{d^2 x^a}{d\tau^2} + \Gamma^a_{bc} \frac{dx^a}{d\tau} \frac{dx^b}{d\tau} = 0$$

which shows that dust particles move along geodesics.

Note: This is one of the assumptions of GR – test particles move on geodesics.

Perfect fluid

In real fluids, the particles interact. Particle scattering provides a pressure on surfaces and interparticle scattering leads to viscosity (friction). The simplest generalisation of relativistic dust is called a "perfect" fluid.

We still neglect viscosity, but account for the pressure p.

The energy-momentum tensor for a perfect fluid can be written

$$T^{ab} = (\rho_0 + p)u^a u^b - p g^{ab}$$

As before, ρ_0 is the proper density and p is the proper pressure.

Note: This is the simplest rank 2 tensor that we can form out of the four velocity and the metric.

In the rest frame (co-moving) we have

$$T^{ab} = (\rho_0 + p)u^a u^b - p\eta^{ab} = \begin{pmatrix} \rho_0 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

Projection

It is useful to note that we can rearrange the terms in the energy-momentum tensor to get

$$T^{ab} = \rho_0 u^a u^b - p\left(g^{ab} - u^a u^b\right) = \rho_0 u^a u^b - p \perp^{ab}$$

This defines the <u>projection</u> $\perp^{ab}$.

What is this good for?

Noting that

$$u_a \perp^{ab} = u_a\left(g^{ab} - u^a u^b\right) = u^b - \underbrace{\left(u_a u^a\right)}_{=1} u^b = 0$$

we see that the pressure term is orthogonal to the four-velocity.

Just like we can use contraction to work out the component along a given four-velocity, we can use the projection to work out the orthogonal piece.

Let's have a look at an example.

Relativistic Euler equations

As in the case of dust, the equations of motion follow from $\nabla_a T^{ab} = 0$

In this case we get

$$\nabla_a T^{ab} = \nabla_a \left[(\rho_0 + p) u^a u^b - p g^{ab} \right] =$$

$$= u^b \nabla_a \left[(\rho_0 + p) u^a \right] + (\rho_0 + p) u^a \nabla_a u^b - g^{ba} \nabla_a p = 0$$

Contract this with u_b to get

$$\underbrace{(u_b u^b)}_{=1} \nabla_a \left[(\rho_0 + p) u^a \right] - \underbrace{u_b g^{ba}}_{=u^a} \nabla_a p = u^a \nabla_a \rho_0 + (p + \rho_0) \nabla_a u^a = 0$$

To see that this is what we expected, we need a little bit of thermodynamics... Assuming that $\rho_0 = \rho_0(n)$, we have the chemical potential

$$\mu = \frac{\partial \rho}{\partial n} \quad \text{and the identity} \quad \mu n = p + \rho_0 \quad \Rightarrow$$

$$\mu u^a \nabla_a n + \mu n \nabla_a u^a = \mu \nabla_a \left(n u^a \right) = 0 \quad \Rightarrow \quad \nabla_a n^a = 0$$

The particle flux is conserved.

Let us now consider the projection orthogonal to the four velocity;

$$\perp_{cb} \nabla_a T^{ab} = \underbrace{\perp_{cb} u^b}_{=0} \nabla_a \left[(\rho_0 + p) u^a \right] + \perp_{cb} (\rho_0 + p) u^a \nabla_a u^b - \underbrace{\perp_{cb} g^{ba}}_{=\perp_c^a} \nabla_a p = 0$$

Introducing the four acceleration

$$a^a = u^b \nabla_b u^a$$

and noting that

$$\perp_{cb} u^a \nabla_a u^b = (g_{cb} - u_c u_b) u^a \nabla_a u^b = u^a \nabla_a u_c$$

we can write the equation in a form that reminds us of Newton's second law;

$$(\rho_0 + p) a_c = \perp_c^a \nabla_a p$$

Pressure gradients drive changes in the four-acceleration.

Perfect fluids do not "move" on geodesics.

Note: The derivative $\perp_c^a \nabla_a$ is purely spatial.