The University Of Sheffield.

DEPARTMENT OF PHYSICS & ASTRONOMY

Autumn Semester 2006-2007

ADVANCED QUANTUM MECHANICS **2 Hours**

Answer THREE questions.

A formula sheet and table of physical constants is attached to this paper.

All questions are marked out of ten. The breakdown on the right-hand side of the paper is meant as a guide to the marks that can be obtained from each part.

1 (a) Find the eigenvalues and eigenfunctions of the following matrix, assuming that α and β are real variables: [3]

$$A = \begin{bmatrix} \alpha & \beta \\ \beta & -\alpha \end{bmatrix}.$$

(b) Construct the matrix S so that its rows will be the eigenfunctions of the matrix A and prove that $SS^T = 1$. [2]

(c) Show by explicit calculations that

$$A' = SAS^T = \begin{bmatrix} \varepsilon & 0 \\ 0 & -\varepsilon \end{bmatrix},$$

where $\pm\varepsilon$ are the eigenvalues. [3]

(d) Provide a "physical'' argument why you would have expected a diagonal form like the above for this matrix. [2]

2 (a) For a one-dimensional harmonic oscillator with the Hamiltonian $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$, find the matrix representation for a, $a^\dagger$, x, and p in the basis of the eigenstates $|n\rangle$ defined via $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$ and $a|n\rangle = \sqrt{n}|n-1\rangle$. [4]

Reminder: $a = \frac{1}{\sqrt{2m\omega\hbar}}(m\omega x + i\,p)$.

(b) Confirm that the matrix representation for $a^\dagger$ is the adjoint of a. Are the matrix representations of x and p Hermitian, and if so, why? [1]

(c) By using the representation of the annihilation operator a in $x-$space, and the definition of the ground-state $a\,\psi_0(x) = 0$, find the normalised ground-state wave function $\psi_0(x)$. [3]

(d) Find the normalised first excited state $\psi_1(x)$ by operating $a^\dagger$ (in differential form) on $\psi_0(x)$. [2]

3 Suppose that the density operator for a quantum system is defined as

$$\hat{\rho}(t) = \sum_{\alpha} |\psi_{\alpha}(t)\rangle P_{\alpha} \langle\psi_{\alpha}(t)|,$$

where P_{α} is the probability of the system being in the state $|\psi_{\alpha}(t)\rangle$.

(a) Show that:

(i) $Tr(\hat{\rho}) = 1$, [2]

(ii) $\langle\hat{K}\rangle = Tr(\hat{\rho}\hat{K})$, [2]

(iii) the equation of motion satisfies $i\hbar\dfrac{\partial\hat{\rho}(t)}{\partial t} = [\hat{H}, \hat{\rho}(t)]$. [4]

(b) If the system is in equilibrium and part of a canonical ensemble, what is P_{α} and what is operator form for the density? [2]

4 Consider a spin-1/2 system with a spin $\vec{S}$ and a magnetic moment $\vec{\mu} = \gamma\vec{S}$ interacting with a time-dependent magnetic field $\vec{B}(t) = B_1(\hat{i}\cos\omega t + \hat{j}\sin\omega t) + B_0\hat{k}$. The spin is defined as $\vec{S} = \dfrac{\hbar}{2}\vec{\sigma}$ in terms of the Pauli matrices $\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, defined in the up/down basis $\{|\uparrow\rangle, |\downarrow\rangle\}$, and the Hamiltonian of the system is defined as $H = -\vec{\mu}.\vec{B}$.

(a) Construct the Hamiltonian in the basis $\{|\uparrow\rangle, |\downarrow\rangle\}$, and decompose it into the diagonal part H_0 and the non-diagonal part W as $H = H_0 + W(t)$. [4]

(b) Defining a general state as $|\psi(t)\rangle = a_1(t)|\uparrow\rangle + a_2(t)|\downarrow\rangle$, write down the Schrödinger equation for this system and extract from it two equations for $a_1(t)$ and $a_2(t)$. Assuming the frequency is set to the resonance $\omega = \gamma B_0$, solve these equations with the initial condition that the spin is up at $t = 0$. [4]

(c) What is the probability that the spin is up at a given time t? [2]

5 The quantum mechanical propagator, G, relates the wave function at any given position and time to the wave function in a prior time and other points in space via

$$\psi(\vec{r}_2,t_2) = \int \mathrm{d}^3\vec{r}_1\ G(\vec{r}_2,t_2;\vec{r}_1,t_1)\psi(\vec{r}_1,t_1)\,.$$

(a) Using the Schrödinger equation $i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle$, find the evolution operator $\hat{U}(t_2,t_1)$ that takes the system from time t_1 to t_2 via $|\psi(t_2)\rangle = \hat{U}(t_2,t_1)|\psi(t_1)\rangle$. [2]
Hint: make sure that causality is guaranteed in the result.

(b) Show that $G(\vec{r}_2,t_2;\vec{r}_1,t_1) = \langle\vec{r}_2|\hat{U}(t_2,t_1)|\vec{r}_1\rangle$, and use the explicit form for the evolution operator found above to write an expression for the propagator in terms of the Hamiltonian. [3]

(c) Show that the Fourier transform

$$G(\vec{r}_2,\vec{r}_1;E) = \int \mathrm{d}t\,G(\vec{r}_2,t;\vec{r}_1,0)\exp[iEt/\hbar]$$

is given by

$$G(\vec{r}_2,\vec{r}_1;E) = \frac{\hbar}{i}\sum_n \frac{\psi_n^*(\vec{r}_1)\psi_n(\vec{r}_2)}{E-E_n+i\delta}\,,$$

where ψ_n are the eigenfunctions of the Hamiltonian with eigenvalues E_n, and δ is an infinitesimal positive number. [3]

(d) Using the above expression for the propagator, show that it satisfies $\frac{i}{\hbar}(E-\hat{H})G(\vec{r}_2,\vec{r}_1;E) = \delta^3(\vec{r}_2-\vec{r}_1)$. [2]

END OF QUESTION PAPER