The University Of Sheffield.

DEPARTMENT OF PHYSICS & ASTRONOMY

Autumn Semester 2006-2007

QUANTUM MECHANICS IN PARTICLE PHYSICS **2 Hours**

Answer THREE QUESTIONS.

A formula sheet and table of physical constants is attached to this paper.

All questions are marked out of ten. The breakdown on the right-hand side of the paper is meant as a guide to the marks that can be obtained from each part.

Note: $\hbar = c = 1$ throughout this paper.

1 (a) What distinguishes an 'active' transformation from a 'passive' transformation? **[1]**

(b) A quantum mechanical process takes a system from an initial state $|a\rangle$ to a final state $|b\rangle$ in some reference frame. This process is also observed in a new frame such that here the states are given by $|a'\rangle$ and $|b'\rangle$. States in these frames are related by a transformation operator $\hat{U}$. Show that $\hat{U}$ is unitary. **[2]**

(c) If $\hat{U}$ corresponds to a continuous transformation it can be written in terms of its generator $\hat{G}$ as

$$\hat{U} = \exp(i\alpha\hat{G}),$$

where α is a real parameter. Show that $\hat{G}$ is hermitian. You need not provide a rigorous proof. **[1]**

(d) Show that the generator of space rotations ϕ about the z-axis is the z-component of the angular momentum operator $\hat{J}_z$ given in polar coordinates by

$$\hat{J}_z = -i\frac{\partial}{\partial\phi}.$$

[3]

(e) Rotational transformations about two different coordinate axes in 3 dimensional space do not in general commute. Explain why this prevents the labelling of quantum mechanical states with eigenvalues of two operators representing angular momentum about two such coordinate axes. **[3]**

2 Spin operators $\hat{S}_i$ for non-relativistic spin-1/2 objects can be represented by $\hat{S}_i = \frac{1}{2}\sigma_i$ where the σ_i are the 2×2 Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(a) Show that the Pauli matrices obey the following commutation and anti-commutation relations for cyclic permutations of *j*, *k* and *l* only:

$$[\sigma_j, \sigma_k] = 2i\varepsilon_{jkl}\sigma_l,\ \{\sigma_j, \sigma_k\} = 2\delta_{jk} I .$$

Here I is the 2×2 Identity matrix and ε_{jkl} is the totally anti-symmetric tensor, given by

$$\varepsilon_{jkl} = 1\ (j, k, l \text{ cyclic}, j \neq k \neq l),$$

$$\varepsilon_{jkl} = -1\ (j, k, l \text{ non-cyclic}, j \neq k \neq l),$$

$$\varepsilon_{jkl} = 0\ \text{(otherwise)}.$$

[3]

(b) Write down the exponential form for the transformation operator $\hat{U}_z(\theta)$ for rotation of a two-component spinor through an angle θ about the z-axis. Use the results from part (a) to show that $\hat{U}_z(\theta)$ can be written as

$$\hat{U}_z(\theta) = \cos(\theta/2) + i\sigma_z \sin(\theta/2) .$$

[4]

(c) What is the effect of a rotation through 360° on a spinor wavefunction?

[1]

(d) The form of the spin operators for spinor solutions to the Dirac equation is somewhat different from that given above. In the Dirac representation for example the operators are given by

$$\hat{S}_i = \frac{1}{2}\begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}.$$

Explain, with reference to the physical interpretation of solutions to the Dirac equation, the need for this alternative definition.

[2]

3 (a) Write down the Klein-Gordon equation for a spin-0 particle of mass m and explain how its form is dictated by relativistic energy conservation. **[2]**

(b) Hence or otherwise derive relationships satisfied by the matrices α_i ($i=1,2,3$) and β appearing in the Dirac equation,

$$i\frac{\partial}{\partial t}\psi = (\boldsymbol{\alpha}\cdot\mathbf{p}+\beta m)\psi\,.$$

[2]

(c) Show how the relationships derived in (b) dictate the form of the solutions to the Dirac equation. Show that some of these solutions must have negative energy and discuss how their interpretation leads to the concept of antiparticles. Explain further how spin-1/2 objects arise naturally from the equation. **[3]**

(d) Show how the Dirac equation may be written in manifestly covariant form, deriving expressions for the Dirac gamma matrices γ_μ in terms of the α_i and β matrices. **[3]**

4 (a) Write down Lagrange's equation for a system of n coordinates $q_i\,(i=1...n)$ and define the Lagrangian L in terms of the kinetic energy T and potential energy V of the system. **[1]**

(b) Discuss briefly how the Lagrangian approach to classical mechanics may be modified to describe classical field theory. **[1]**

(c) Using the explicit form for $F_{\mu\nu}$ in terms of **E** and **B**, namely

$$F_{\mu\nu}=\begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix},$$

show that

$$F_{\mu\nu}F^{\mu\nu}=2(\mathbf{B}^2-\mathbf{E}^2).$$

[2]

(d) Using the definitions of **E** and **B** in terms of the electromagnetic scalar and vector potentials V and $\mathbf{A}$,

$$\mathbf{E}=-\nabla V-\frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B}=\nabla\times\mathbf{A},$$

cast the expression for $F_{\mu\nu}F^{\mu\nu}$ from part (c) in terms of components of the 4-vector potential A_μ, where

$$A_\mu=(V,-\mathbf{A}).$$

[3]

(e) Hence use the Euler-Lagrange equation for the A_μ field,

$$\frac{\partial \mathcal{L}}{\partial A_\mu}-\sum_\alpha \partial_\alpha \frac{\partial \mathcal{L}}{\partial\left(\partial_\alpha A_\mu\right)}=0,$$

together with the QED Lagrange density,

$$\mathcal{L}_{QED}=\overline{\psi}(i\gamma^\mu\partial_\mu-m)\psi-J^\mu A_\mu-\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu},$$

where $J^\mu=q\overline{\psi}\gamma^\mu\psi=(\rho,\mathbf{J})$, to derive the inhomogeneous Maxwell equations

$$\nabla.\mathbf{E}=\rho, \qquad \nabla\times\mathbf{B}-\frac{\partial \mathbf{E}}{\partial t}=\mathbf{J},$$

for the time-like and one space-like component of J^μ. **[3]**

5 (a) Write down the four criteria for a set of objects (together with an operation ‘o’) to be considered a ‘group’. **[2]**

(b) Write down the mathematical definition of each of the following terms used to describe groups:

(i) non-Abelian,

(ii) Special Unitary,

(iii) Special Orthogonal,

(iv) Lie.

Give one example of a non-Abelian Special Orthogonal Lie group from geometry, explaining what it represents. **[3]**

(c) QED is an example of a U(1) Abelian gauge theory. The QED Lagrange density can be written in terms of the electron field ψ and the photon field A_μ as

$$\mathcal{L}_{QED} = \overline{\psi}(i\gamma^\mu \partial_\mu - m)\psi - q\overline{\psi}\gamma^\mu A_\mu \psi - \tfrac{1}{4} F^{\mu\nu} F_{\mu\nu},$$

where the field strength tensor $F_{\mu\nu}$ is defined by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu .$$

(i) Quantum Chromo-Dynamics (QCD) is an example of a non-Abelian SU(3) gauge theory and describes the propagation and interactions of quark and gluon fields. By analogy with QED write down the Lagrange density for QCD in terms of the quark and gluon fields Q_i and g^a_μ $(a = 1...8)$, the gluon field strength tensor $G^a_{\mu\nu}$ and the generators of the SU(3) gauge group T_a. Write down also an expression for $G^a_{\mu\nu}$ in terms of the gluon fields, commenting on any differences between this and the expression for $F_{\mu\nu}$ given above. **[2]**

(ii) What is the main difference between the interactions of gauge bosons in QED and QCD? How does this relate to the structure of their gauge groups? Draw Feynman diagrams to illustrate your answer. *Hint: refer to your answer to part c(i).* **[3]**

END OF EXAMINATION PAPER