

## DEPARTMENT OF PHYSICS \& ASTRONOMY

Spring Semester 2006-2007
STATISTICAL PHYSICS
2 Hours

Answer THREE questions.
A formula sheet and table of physical constants is attached to this paper.
All questions are marked out of ten. The breakdown on the right-hand side of the paper is meant as a guide to the marks that can be obtained from each part.

1 Consider a solid made up of $N$ classical magnetic dipoles with moment $\vec{\mu}$ in an external magnetic field $\vec{B}$. The energy of a dipole in the external magnetic field is given as:

$$
E=-\vec{\mu} \cdot \vec{B}=-\mu B \cos \theta,
$$

where $\theta$ is the angle between the direction of the dipole and the magnetic field.
(a) Assuming that the dipoles do not interact with each other, show that the partition function and the thermodynamic properties of the system can be obtained using the single-particle partition function.
(b) The partition function for one dipole is given by

$$
Z=\int \frac{d \Omega}{4 \pi} e^{-\beta E}
$$

where $\beta=1 / k_{\mathrm{B}} T$, and the integration is performed over the solid angle $\Omega$ that defines the direction of the dipole in 3D space. Using

$$
\int \frac{d \Omega}{4 \pi}=\frac{1}{4 \pi} \int_{0}^{2 \pi} d \phi \int_{0}^{\pi} \sin \theta d \theta=\frac{1}{4 \pi} \int_{0}^{2 \pi} d \phi \int_{-1}^{1} d x
$$

where $\theta$ and $\phi$ are the angles in the spherical coordinates (assuming that $\vec{B}$ is in the $z$-direction) and $x=\cos \theta$, show that the partition function is given by

$$
\begin{equation*}
Z=\frac{\sinh (\beta \mu B)}{\beta \mu B} \tag{3}
\end{equation*}
$$

(c) Show that the average energy is given by

$$
U=-\mu B\left[\operatorname{coth}(\beta \mu B)-\frac{1}{\beta \mu B}\right] .
$$

Examine the limiting behaviours of this expression at high and low temperatures. Sketch the energy as a function of temperature.
Hint: use coth $x-\frac{1}{x} \approx \frac{1}{3} x$ for $x \ll 1$.
(d) Calculate the heat capacity $C_{V}=\partial U / \partial T$, and sketch it as a function of temperature.

2 A container of volume $V$ is filled with a mixture of two ideal gases of type A and B. There are $N_{\mathrm{A}}$ indistinguishable molecules of type A with mass $m_{\mathrm{A}}$ and $N_{\mathrm{B}}$ indistinguishable molecules of type B with mass $m_{\mathrm{B}}$, while the two types can be clearly distinguished from one another.
(a) How do we count the number of states in the phase space for a gas of particles? Derive the single-particle partition function for the gas particles, using the sum over configuration space.
(b) Write an expression for the many-particle partition function of the system in terms of the relevant single-particle partition functions.
(c) Using the partition function, calculate the total Helmholtz free energy $F$, the internal energy, the entropy, and the equation of state of the system.
(d) Which of the thermodynamic quantities calculated above are the same as those of a homogenous ideal gas of $N=N_{\mathrm{A}}+N_{\mathrm{B}}$ particles, and which are not? Explain why.

3 (a) Show that the average internal energy of a system with a constant volume and particle number in thermal contact with a heat bath at a temperature $T$ is

$$
U=-\frac{\partial \ln Z}{\partial \beta}
$$

where $Z$ is the canonical partition function and $\beta=1 / k_{\mathrm{B}} T$.
(b) From quantum mechanics, the rotational energy levels of a molecule are

$$
\varepsilon_{j}=\frac{\hbar^{2}}{2 I} j(j+1),
$$

where $I$ is the moment of inertia and the angular momentum quantum number $j$ takes on integer values $(j=0,1,2, \cdots)$. The degeneracy of the $j^{\text {th }}$ level is $g_{j}=2 j+1$.
(i) Write down an expression for the partition function for the rotational motion.
(ii) For $T \ll T_{r}$, where $T_{r}=\hbar^{2} /\left(2 I k_{\mathrm{B}}\right)$, the sum may be truncated after $j=1$. Show that in this limit the contribution of the rotational motion to the heat capacity reads

$$
\begin{equation*}
C_{r}=3 k_{\mathrm{B}}\left(\frac{2 T_{r}}{T}\right)^{2} e^{-2 T_{r} / T} . \tag{3}
\end{equation*}
$$

Hint: use $\ln (1+x) \approx x$ for $x \ll 1$.
(iii) Explain why the sum over $j$ in the partition function may be converted into an integral in the high temperature limit, over a new variable $y=\left(j+\frac{1}{2}\right)^{2} \hbar^{2} /\left(2 I k_{\mathrm{B}} T\right)$. Hence, show that in this
limit the heat capacity satisfies the equipartition theorem.
Hint: use $\Delta y=2 \Delta j\left(j+\frac{1}{2}\right) \hbar^{2} /\left(2 I k_{\mathrm{B}} T\right)=\left(j+\frac{1}{2}\right) \hbar^{2} /\left(I k_{\mathrm{B}} T\right)$
as $\Delta j=1$.

4 An ideal gas of $N$ molecules of mass $m$ is contained in a cylinder of length $L$ and radius $R$. The cylindrical container is rotating about its axis at an angular velocity $\omega$, and is at equilibrium with temperature $T$.
(a) What is the effect of rotation on the density profile of the gas? Make a sketch of this profile and explain why you expect this behaviour.
(b) Write down the expression for the energy of single-particle states and the Boltzmann distribution for the gas in the rotating frame of the cylinder, assuming that the particles experience an additional centrifugal potential energy of the form

$$
V(r)=-\frac{1}{2} m \omega^{2} r^{2}
$$

where $r$ is the distance from the axis in the cylindrical coordinate system. Separate the translational and interaction parts of the partition function and the Boltzmann distribution, and show that the expression for (the interaction part of) the Boltzmann distribution is consistent with your physical description of the density profile in part (a).
(c) Calculate the partition function and the Helmholtz free energy of the gas in the rotating frame of the cylinder.
(d) Calculate the limiting form of the free energy for $m \omega^{2} R^{2} \ll k_{\mathrm{B}} T$. Hint: use $e^{x} \approx 1+x+x^{2} / 2$ and $\ln (1+x) \approx x$ for $x \ll 1$.
(e) Using the above result, calculate the pressure that is exerted on the container walls, by considering an infinitesimal increase in the radius of the container. Check that you can reproduce the ideal gas pressure.

5 Consider a gas of $N$ fermions that are restricted to move only in a onedimensional space of size $L$, in the thermodynamic limit.
(a) Describe why it is permissible to use a grand canonical ensemble for this system with fixed number of particles. Write down the Fermi-Dirac distribution for the average occupation number of fermions with energy $\varepsilon$ and chemical potential $\mu(T)$.
(b) Write down the dispersion relation of the particles. What is the difference between this result and the dispersion relation of the 3D Fermi gas?
(c) By counting the number of states in the wave-vector space, calculate the Fermi wave-vector $k_{\mathrm{F}}$ and the Fermi energy $\varepsilon_{\mathrm{F}}$.
(d) Using the density of states in the wave-vector space, calculate the density of states $g(\varepsilon)$.
(e) Write down the equation that determines the chemical potential, and (by explicit calculations) show that it is consistent with $\mu(T=0)=\varepsilon_{\mathrm{F}}$.
(f) Calculate the average kinetic energy and the one-dimensional pressure (the equivalent of 3D pressure adapted to this case) at $T=0$.

## END OF QUESTION PAPER

