The University Of Sheffield.

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DEPARTMENT OF PHYSICS AND ASTRONOMY

Spring Semester 2006-2007 2 Hours

DARK MATTER AND THE UNIVERSE

Answer THREE questions including question 1.

A formula sheet and table of physical constants is attached to this paper.

All questions are marked out of ten. The breakdown on the right-hand side of the paper is menat as a guide to the marks that can be obtained from each part.

1 (a) Define 'cold dark matter'. ***(1 mark)***

(b) Give one example of a dark matter candidate that is typically not cold. ***(1 mark)***

(c) What is the de Broglie wavelength of a particle of rest energy 2 MeV and total energy 5 MeV ? ***(2 marks)***

(d) Define 'baryonic dark matter'. ***(1 mark)***

(e) Give an experimental technique suitable for searching for baryonic dark matter candidates in the Milky Way. ***(1 mark)***

Primordial nucleosynthesis of the lightest elements gives us an experimental probe of the abundance of baryonic dark matter.

(f) Why are quasars used as light sources for probing the abundance of hydrogen in today's universe ? ***(2 marks)***

(g) What is the connection between currently observable light element abundances and the baryonic dark matter fraction ? ***(2 marks)***

2 Rotation curve measurements galaxy NGC4569 yield the results shown in Figure 1. This data does not follow the usual pattern as beyond a radius of 0.7 kpc the rotation velocity v as a function of radius r is well described by the relation

$$v(r) = v_0 + Ar,$$

where v_0 and A are constants.

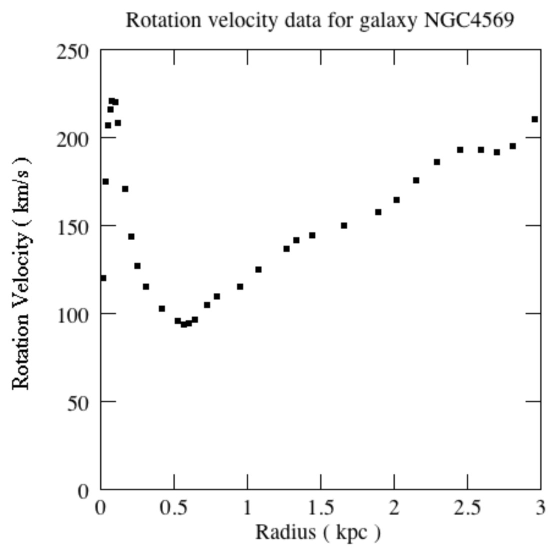

Figure 1: Velocity of rotation for a range of radii about the centre of spiral galaxy NGC4569

(a) Estimate the constants v_0 and A based on the data points having $r > 0.7$ kpc. ***(2 marks)***

(b) Starting with this model for $v(r)$, show that the total mass $M(r)$ inside radius r is

$$M(r) = \frac{1}{G}(Jr^3 + Kr^2 + Lr),$$

where J, K, and L are constant functions of v_0 and A, and G is Newton's gravitational constant. Give numerical values for J, K and L. Neglect the effects of the core and assume a spherically symmetric dark matter distribution. ***(5 marks)***

(c) Derive an expression for the density $\rho(r)$ of dark matter at radius r outside the core in terms of A, v_0, r and G. Show that one component of the matter distribution is of uniform density, and give a numerical value for the density of this component. ***(3 marks)***

3 Figure 2 shows an image of a galaxy-lens pair from the Sloan digital sky survey, and a spectrum from the circular arc surrounding the lensing galaxy. Several emission lines are shown, amongst them an oxygen line, [OIII], which has a measured lab wavelength of 5007 Å. The image is 8 arcseconds on a side.

(a) Give three applications of gravitational lensing to searches for dark matter **(3 marks)**

(b) Estimate the redshift of the background source. **(1 mark)**

(c) The wavelength of the [OIII] line in the lensing object as measured by an observer on Earth is found to be 6620Å. Calculate the distance between the lens and the source in Mpc, and estimate the mass of the lens in solar masses. **(3 marks)**

(d) What is the total angle through which light is bent from its initial trajectory during emission from the source to its final trajectory entering the telescope? **(3 marks)**

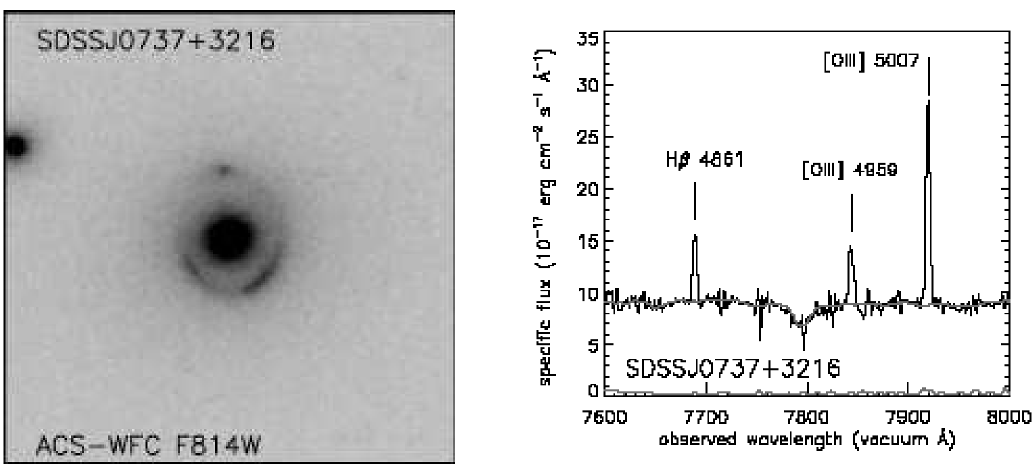

Figure 2: An image from the Sloan digital sky survey, and a spectrum of the ring shaped emission region surrounding the lensing galaxy.

4 (a) What theoretical considerations disfavour axions of rest energies less than 1 μeV? ***(1 mark)***

(b) What experimental considerations disfavour axions of rest energies between 1 meV and 10 keV? ***(1 mark)***

(c) What experimental considerations disfavour axions of rest energies above 10 keV? ***(1 mark)***

(d) A resonant cavity axion detector consists of two cavities each having a resonant frequency of 1.2 GHz. Using electric field probes, each cavity is critically coupled to a coaxial RF transmission line. The signals from the two cavities are then combined using a matched resistive power combiner, and the combined power is fed to the input of a matched RF amplifier. Axions lead to the generation of continuous RF power in a bandwidth $B = 200$ Hz about 1.2 GHz in each cavity. The axion signal power converted in each cavity is $P_a = 10^{-22}$ W. Each cavity is kept at a physical temperature T_C of 2 K.

(i) What is meant by critically coupled? ***(1 mark)***

(ii) The noise temperature of the amplifier is $T_A = 4.6$ K. What is the total noise power over the axion signal bandwidth at the amplifier input? ***(2 marks)***

(iii) What is the total axion signal power at the amplifier input? Assume that the power combiner succeeds in adding the signals from the two cavities in phase, with no power losses. ***(1 mark)***

(iv) Write down an algebraic expression for the integration time to achieve an axion signal power to noise power fluctuation ratio of S. Express your answer as a function of T_C, T_A, P_a,B, S, and Boltzmann's constant k_B. ***(2 marks)***

(v) How many seconds of averaging would be required to achieve $S = 4$? ***(1 mark)***

5 Assume for this question that WIMP dark matter consists of an isothermal sphere of neutralinos each of rest energy 830 GeV and having an r.m.s. velocity of 270 km s^{-1}.

(a) The WIMP nucleus cross section is proportional to A^2. Explain the origin of the two multiplicative factors of A. ***(3 marks)***

(b) Calculate the maximum kinetic energy that could be imparted to a single atom of xenon, atomic mass 131, in a collision with a single WIMP of the given rest energy and having a velocity equal to the r.m.s. velocity in the local halo. ***(4 marks)***

(c) An 800 kg argon detector contains 1 nucleus in 10^{11} of ^{39}Ar, which has a half life of 256 years. Estimate the background rate in the argon detector from these decays, assuming they are detected with 100% efficiency. ***(1 mark)***

(d) The data acquisition system attached to this detector records 100 μs of data for each radioactive decay acquired. What data analysis problems do you think this background might cause? ***(2 marks)***

End of Question Paper

1 (a) Dark matter that was non relativistic ($\frac{1}{2}$) at the time of structure formation ($\frac{1}{2}$)

(b) Neutrino dark matter (1)

(c) $\lambda = \frac{h}{p} = \frac{2\pi\hbar c}{pc} = \frac{2\pi \times 0.2 \text{ GeV fm}}{\sqrt{E^2 - m^2c^4}} = \frac{2\pi \times 0.2 \text{ GeV fm}}{\sqrt{5^2 - 2^2} \times 10^{-3} \text{GeV}} = 274 \text{ fm}$

Accept 250 – 300 fm (2)

(d) Matter consisting of protons & neutrons, the stable baryons (1)

(e) Searching for microlensing of starlight by dark, low luminosity stars (1)

(f) They are very high redshift, and hence very distant objects (1) having strong atomic hydrogen emission lines, particularly the Lyman alpha line (1)

(g) Current light element abundances relative to Hydrogen are the same except for Lithium as those immediately after primordial nucleosynthesis. (1) These can be predicted knowing the mass to light ratio at nucleosynthesis which is a measure of the baryonic dark matter contribution today. (1)
Match the observed relative abundances of the light elements to those predicted as a function of the baryon fraction.

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2 (a) $V_0 = 70$ km/s accept 60 – 75 (1)

$A = \frac{230 - 60}{3}$ km/s/kpc $= 57 \text{ km s}^{-1} \text{ kpc}^{-1}$ (1)

(b) $v(r) = v_0 + Ar$

$\frac{GM(r)}{r^2} = \frac{v(r)^2}{r}$ $M(r) = \frac{1}{G} r v(r)^2 = \frac{1}{G} r (v_0 + Ar)^2$ (2)

$= \frac{1}{G} r (v_0^2 + 2Av_0 r + A^2 r^2)$

$M(r) = \frac{1}{G}(A^2 r^3 + 2Av_0 r^2 + v_0^2 r)$ (1)

$J = A^2$ $K = 2AV_0$ $L = V_0^2$ $\frac{1}{2}$

$J = 3250 \text{ km}^2 \text{s}^{-2} \text{kpc}^{-2}$
$K = 7980 \text{ km}^2 \text{s}^{-2} \text{kpc}^{-1}$
$L = 4900 \text{ km}^2 \text{s}^{-2}$
$\frac{1}{2} + \frac{1}{2} + \frac{1}{2}$

(c) $dM(r) = 4\pi r^2 \rho(r)\, dr$

$$\rho(r) = \frac{1}{4\pi r^2} \frac{dM(r)}{dr}$$

$$= \frac{1}{4\pi r^2} \frac{1}{G} \left(3A^2 r^2 + 4 A v_0 r + v_0^2\right)$$

$$\rho(r) = \frac{3A^2}{4\pi G} + \frac{A v_0}{\pi G r} + \frac{v_0^2}{4\pi r^2 G}$$

Uniform density component has density $\frac{3A^2}{4\pi G}$

$A = 57 \text{ km s}^{-1} \text{ kPc}^{-1}$

$$= 57 \text{ km s}^{-1} \text{ km}^{-1} \times \frac{1}{3.1 \times 10^{16}} \frac{\text{km}}{\text{kPc}}$$

$$= 1.9 \times 10^{-15} \text{ s}^{-1}$$

$$\frac{3A^2}{4\pi G} = \frac{3 \times (1.9 \times 10^{-15})^2 \text{ s}^{-2}}{4\pi \cdot 6.7 \times 10^{-11} \text{ kg}^{-1} \text{ m}^3 \text{ s}^{-2}} = 1.3 \times 10^{-20} \text{ kg/m}^3$$

$$= 1.2 \times 10^{-9} \text{ J/cm}^3$$

$$= 7.2 \text{ GeV/cc}$$

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3 (a) Brown dwarfs, baryonic dark stars in galaxies (1)
Reconstruction of the dark matter contents of a galaxy cluster (1)
Large area surveys for dark matter density determination using very large numbers of spectroscopically identified lensed sources (1)

(b) $z_s = \frac{\Delta\lambda}{\lambda_e} = \frac{7925 - 5007}{5007} = 0.58$ (1)

(c) $z_L = \frac{~~5985~~\ 6220 - 5007}{5007} = 0.32$ (1) $z_L = 0.3223$ $\frac{H_0 d_L}{c} = z + \frac{1}{2}(1 - q_0) z^2$

$z_s + \frac{1}{2}(1.2) z_s^2 = 0.78$

$d_s = \left(\frac{3 \times 10^5 \text{ km s}^{-1}}{72 \text{ km s}^{-1} \text{ MPc}^{-1}}\right) \cdot 0.78 = 3250 \text{ MPc}$ luminosity distance of source

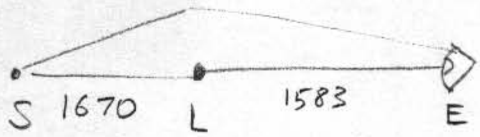
$z_L + \frac{1}{2}(1.2) z_L^2 = 0.38$

$d_L = \left(\frac{3 \times 10^5 \text{ km s}^{-1}}{72 \text{ km s}^{-1} \text{ MPc}^{-1}}\right) \cdot 0.38 = 1583 \text{ MPc}$ luminosity distance of lens

distance between source and lens is 1670 MPc

3c, continued.

$$\alpha = \sqrt{\frac{4MG\,D_{SL}}{c^2 D_{LE}(D_{SL}+D_{LE})}}$$



$$\frac{D_{SL}}{D_{LE}(D_{SL}+D_{LE})} = \frac{1670}{1583(3250)} = 3\cdot25\times10^{-4}\ \mathrm{MPc}^{-1}$$

$$\frac{3\cdot25\times10^{-4}}{3\cdot1\times10^{22}}\ \mathrm{m}^{-1}$$

$$= 10^{-26}\ \mathrm{m}^{-1}$$

$$\alpha = \sqrt{\frac{4\times M\times 6\cdot7\times10^{-11}\ \mathrm{kg^{-1}m^3s^{-2}}\times10^{-26}\ \mathrm{m^{-1}}}{(3\times10^8)^2\ \mathrm{m^2s^{-2}}}}$$

$$\alpha = \sqrt{M}\cdot\sqrt{3\times10^{-53}}$$

$$= 5\cdot4\times10^{-27}\sqrt{M}$$

$$\alpha = \frac{1}{5}\times 8\ \text{arc seconds}$$

$$= 7\cdot8\times10^{-6}\ \text{rad}$$

$$7\cdot8\times10^{-6}\ \text{rad} = 5\cdot4\times10^{-27}\sqrt{M}$$

$$M = 2\times10^{42}\ \mathrm{kg}$$

$$M = 10^{12}M_\odot$$

(d) $$\Delta\Phi = \frac{4MG}{c^2 b} = \frac{4\times2\times10^{42}\ \mathrm{kg}\times6\cdot7\times10^{-11}\ \mathrm{kg^{-1}m^3s^{-2}}}{(3\times10^8)^2\times((1583\times3\cdot1\times10^{22})\times7\cdot8\times10^{-6})}$$

$= 1\cdot6\times10^{-5}$ rad $= 3\cdot2$ arc seconds or just notice that $D_{SL} \simeq D_{LE}$ & double the angle.

(e) $d_{LE} = 1583\times2 = 3166$ MPc

$$\frac{1670}{1583\times3250} \rightarrow \frac{1670}{3166(3250+1583)} \qquad 3\cdot25\times10^{-4} \rightarrow 1\times10^{-4}$$

take previous angle and multiply by $\sqrt{\frac{1}{3\cdot25}} = 0\cdot55$

radius of Einstein ring now is $0\cdot55\times 8/5$ arc seconds, or $0\cdot9$ arc secs

4 (a) Overclosure of the Universe (1)

(b) Energy transport by axions would cause red giants to cool too rapidly, Or SN1987A pulse would be shortened (1)

(c) Non-observation of axions in particle accelerators (1)

(d) Half the power in a resonant line is dissipated in the coupling circuit or the width of the resonant line is doubled compared to the weakly coupled case. (1)

(e) $P = k_B T B = 1.38 \times 10^{-23} \, JK^{-1} \cdot (2 + 4.6)K \cdot 200Hz$

$= 1.8 \times 10^{-20} \, W.$

(f) 10^{-22} W per cavity, but half the power is lost in the cavity walls, so power towards cavity is 0.5×10^{-22} W from each cavity, or 10^{-22} W overall (2)

(g)
$$S = \frac{P_S}{P_N} \sqrt{Bt}$$

$$P_S = 2P_a$$

$$P_N = k_B (T_C + T_A) B$$

$$S = \frac{2P_a}{k_B (T_C + T_A) B} \sqrt{Bt}$$

$$\frac{S k_B (T_C + T_A)}{2P_a} = \sqrt{\frac{t}{B}}$$

$$t = \frac{B S^2 k_B^2 (T_C + T_A)^2}{4 P_a^2}$$ (2)

(h)
$$t = \frac{200 \times 4^2 \times (1.38 \times 10^{-23})^2 \times 6.6^2}{4 \times (10^{-22})^2}$$

$= 663$ seconds

11 minutes (1)

5 a) - Coherence of scattering off multiple nucleons in the nucleus. Without coherence,

$$|\psi|^2 = \sum_i^A |\psi_i|^2 = A|\psi_i|^2$$

↑ amplitude from A nucleons

With coherence,

$$|\psi|^2 = \left| \sum_i^A \psi_i \right|^2 = A^2|\psi_i|^2$$ so factor of A from coherence (1)

- Enhanced density of final states in energy space

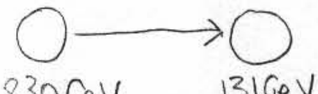
At fixed energy

Nucleon alone has momentum $\sqrt{2mE}$

Whole nucleus has momentum $\sqrt{A} \cdot \sqrt{2mE}$ $m \to Am$

density of states $\propto p^2 dp$

p^2 gets a factor of A so density of final states enhanced by A (1)

b) 

max energy transfer in direct hit

energy conservation $\frac{1}{2}m_w u^2 = \frac{1}{2}m_w v_w^2 + \frac{1}{2}m_n v_n^2$

$$m_w u^2 = m_w v_w^2 + m_n v_n^2$$

momentum conservation $m_w u = m_w v_w + m_n v_n$

$$v_w^2 = \frac{(m_w u - m_n v_n)^2}{m_w^2}$$

$$m_w u^2 = \frac{(m_w u - m_n v_n)^2}{m_w} + m_n v_n^2$$

$$m_w^2 u^2 = m_w^2 u^2 + m_n^2 v_n^2 - 2m_w m_n u v_n + m_n m_w v_n^2$$

$$m_n v_n^2 + m_w v_n^2 = 2m_w u v_n$$

$$(m_w + m_n) v_n = 2m_w u$$

$$v_n = \frac{2m_w u}{(m_w + m_n)} \qquad \frac{1}{2}m_n v_n^2 = \frac{2m_w^2 u^2 m_n}{(m_w + m_n)^2}$$

Maximum kinetic energy of WIMP after collision

$$= 2 \times \left(\frac{M_W}{M_W + M_N}\right)^2 M_N c^2 \left(\frac{u}{c}\right)^2$$

$$= 2 \times \left(\frac{830}{830+131}\right)^2 \times 131 \times \left(\frac{\cancel{200}\ 270}{3\times10^5}\right)^2$$

$$= \cancel{21.7}\ \text{keV} \quad 39.5\ \text{keV}$$

c.) Number of radioactive nuclei

$$= \frac{800}{0.131} \times N_A \times 10^{-11}$$

$$= 3.7\times10^{16} \text{ nuclei}$$

Half life of 256 years

$$N = N_0 e^{-t/\tau} \qquad \frac{1}{2} = e^{-t_{1/2}/\tau}$$

$$\ln 2 = \frac{t_{1/2}}{\tau}$$

$$t_{\frac{1}{2}} = \tau \ln 2$$

$$N = N_0 e^{-\frac{\ln 2\, t}{t_{1/2}}}$$

$$\frac{dN}{dt} = -\frac{N_0 \ln 2}{t_{1/2}} e^{-\frac{\ln 2\, t}{t_{1/2}}} = N \cdot \frac{\ln 2}{t_{1/2}}$$

$$\frac{dN}{dt} = \frac{3.7\times10^{16} \times 0.693}{256 \times 86400 \times 365.25}$$

$$= 3.17 \times 10^6 \text{ decays per second!!}$$

d) Number of decays per record $3.17\times10^6 \times 10^{-4} = 317$ decays per record. Hard to see signals in this forest of background events in each record.

PHY-323 Formula Sheet

Isothermal Sphere Model. Let v be the measured rotation velocity of baryonic material in the galaxy at a distance r from the galactic centre. The total mass $M(r)$ within radius r and the total matter density $\rho(r)$ at radius r are given by

$$M(r) = \frac{v^2 r}{G} \quad \rho(r) = \frac{v^2}{4\pi r^2 G},$$

where $G = 6.7 \times 10^{-11}$ kg^{-1} m^3 s^{-2} is Newton's gravitational constant. Assuming the isothermal sphere model is correct, the energy density in our local dark matter halo is 0.3 GeV/cc.

Virial Theorem. The virial theorem states that for a gas of particles in thermodynamic equilibrium and bound by the action of force whose magnitude drops as $1/r^2$, where r is the distance from the centre of mass of the distribution, the expectation $\overline{T}$ of the total kinetic energy of the gas is related to the expectation $\overline{V}$ of the total potential energy of the gas particles in the potential by $\overline{V} = -2\overline{T}$.

MOND. In the theory of Modified Newtonian Dynamics (MOND), if the predicted Newtonian acceleration a_N of the body is of the order of $a_0 = 1.2 \times 10^{-10}$ ms^{-1} or less, then the observed acceration a is given by

$$a = \sqrt{a_0 a_N}$$

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Gravitational Lensing by Clusters. The angular size α in radians of an Einstein ring measured from the centre of the ring to the circumference is given by

$$\alpha = \sqrt{\frac{4MGD_{SL}}{c^2 D_{LE}(D_{SL} + D_{LE})}}$$

where M is the cluster mass, D_{SL} is the distance from the source to the lensing cluster, D_{LE} is the distance from the cluster to Earth, G is Newton's gravitational constant, and c is the speed of light. All masses, lengths, and physical constants are in SI units.

Hubble's Law Hubble's law relates the distance to an astrophysical object to the redshift of its spectral lines. For nearby objects (objects having redshifts $z = \Delta\nu/\nu < 0.1$), Hubble's law gives a linear relationship between recession velocity and distance of a source, $v = H_0 d_L$. If v is a velocity of recession in km s^{-1} and d_L is a luminosity distance in MPc, then the current best fit value of the Hubble constant is $H_0 = 72$ km s^{-1} MPc^{-1}. Where z is greater than 0.5, the acceleration of the expansion rate needs to be accounted for, so Hubble's law becomes

$$\frac{H_0 d_L}{c} = z + \frac{1}{2}(1 - q_0) z^2,$$

where $q_0 = -0.2$ is the decceleration parameter and d_L is the luminosity distance to the source.

Relativity The total energy E, momentum-energy $E_p = pc$ and mass energy $E_R = mc^2$ of a relativistic particle are related by $E^2 = p^2c^2 + m^2c^4$, where m is the particle mass and p is the particle momentum. The De-Broglie wavelength of a particle or wave is given by $\lambda = h/p$, where p is the momentum. Note that $\hbar c = 0.2$ GeV fm in energy units.

Radioactive Decay and Cross Sections The mean life of a radioactive isotope is the time taken for all but 1/e of the original sample to decay, where e = 2.718. The formula for the number N of undecayed atoms out of an original N_0 at time $t = 0$ is $N = N_0 \exp[-t/\tau]$, where τ is the mean life. The probability of interactions occuring is often give in terms of a cross section for the process. A cross section of 1 barn corresponds to 10^{-28} m^2.

Nuclear Equilibrium For a decay chain where nuclear equilibrium has been reached, and where each nucleus of the $n+1^{\text{th}}$ isotope down the chain results from the decay of a nucleus of the n^{th} isotope, the ratio of the number N_{n+1} of nuclei of the $n+1^{\text{th}}$ isotope to the number N_n of nuclei of the n^{th} isotope is given by

$$\frac{N_{n+1}}{N_n} = \frac{t_{1/2,n+1}}{t_{1/2,n}},$$

where $t_{1/2,n+1}$ is the half life of the $n+1^{\text{th}}$ isotope and $t_{1/2,n}$ is the half life of the n^{th} isotope.

Supersymmetric WIMP Dark Matter Experiments to detect WIMPs try to detect the energy of the recoil of a heavy nucleus with which a single WIMP particle from our galactic halo has collided. If a wimp of velocity v and mass m_W strikes a target nucleus of mass m_T, which then recoils at an angle θ to the direction of incidence of the WIMP, the recoil energy E_R of the nucleus is

$$E_R = \tfrac{\mu^2 v^2}{m_T}(1-\cos\theta) \quad \text{where} \quad \mu = \tfrac{m_T m_W}{m_T + m_W} \;,$$

and m_T and m_W have the units GeV/c^2.

The total event rate for WIMP-nuclear interactions r assuming an isothermal sphere model of dark matter is given by

$$r = \frac{10^{-31} N_A M_D \rho_H v \sigma_0^{\text{pb}} A^2}{m_W c^2},$$

where $N_A = 6 \times 10^{23}$ is Avogadro's number, M_D is the mass of the detector in kilograms, ρ_H is the density of the halo in GeV/cc, v is the WIMP velocity in m/s, σ_0^{pb} is the WIMP-nucleon cross section in picobarns, A is the atomic mass number of the nucleus, and $m_W c^2$ is the rest energy of the WIMP in GeV.

Axion Dark Matter Experiments to detect Axions try to detect the conversion of axions into photons in a resonant cavity.

In the case of a matched circuit, the noise power emitted by a source resistor at temperature T over a bandwidth B is $k_B T B$, where $k_B = 1.3 \times 10^{-23}$ JK^{-1} is Bolzmann's constant.

The radiometer equation governing the signal-to-noise ratio SNR, which is the ratio of the axion signal power to the size of the statistical fluctuations in power due to noise background, is given by

$$\text{SNR} = \frac{\text{P}_\text{S}}{\text{P}_\text{N}} \sqrt{\text{Bt}},$$

where P_S is the axion signal power, P_N is the noise background power, B is the bandwidth over which the signal and noise are spread in frequency (Hz), and t is the time over which the power measurement is made.