The University Of Sheffield.

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DEPARTMENT OF PHYSICS AND ASTRONOMY

Spring Semester 2006-2007 2 Hours

RELATIVITY AND COSMOLOGY

A formula sheet and table of physical constants is attached to this paper. Answer question 1 (COMPULSORY) and two out of the other four questions.

1 COMPULSORY

(a) State Einstein's two postulates of special relativity. ***(2 marks)***

(b) Is the geometry of the surface of a cylinder intrinsically flat, positively curved, or negatively curved? Briefly explain why. ***(2 marks)***

(c) A π^0 meson, rest energy of 135 MeV, decays to two gamma rays

$$\pi^0 \rightarrow 2\gamma.$$

If the π^0 is moving at a velocity of magnitude $v = 0.5c$ in the lab when it decays, what is the maximum energy of an emitted gamma, also in the lab frame? ***(3 marks)***

(d) An object has the same luminosity as the Milky Way, but is at a redshift of $z = 1$. What is the energy flux from this object here on Earth in units of the Milky Way luminosity? ***(3 marks)***

2 A photon is directed vertically downwards in a region where the acceleration due to gravity is g downwards and can be considered uniform. Two observers make measurements on the photon. Observer A is released from rest at $t = 0$ as the photon passes him, and falls freely in the gravitational field. Observer B is at rest with respect to the reference frame of observer A when she was released. At time $t = 0$, both observers measure the wavelength of the photon, and both agree that it is λ.

(a) State Einstein's principle of equivalence. ***(1 mark)***

(b) How does the wavelength of the photon evolve after $t = 0$ in the frame of observer A? Explain your answer. ***(2 marks)***

(c) Assume that Δt is sufficiently small that observer A is always travelling at a velocity much less than that of light with respect to observer B. At time $t = \Delta t$, what is the wavelength of the photon as observed by observer B, to first order in v? Express your answer in terms of v, c, g, λ, and Δt, but not necessarily all of these quantitites. ***(5 marks)***

(d) When the photon has fallen through height h, what is the wavelength as measured by observer B? Express your answer in terms of g, c, and h. ***(2 marks)***

3 The mean energy of photons from the microwave background, corresponding to a temperature around 3 K, is 6×10^{-4} eV. It is expected that in rare processes, high energy cosmic ray protons may interact with these photons to produce a neutron and a charged π meson:

$$p^+ + \gamma \rightarrow n + \pi^+.$$

This question concerns the threshold proton energy for such a collision, known as the GZK cutoff.

(a) State the law of conservation of four momentum. ***(1 mark)***

(b) Working in the centre of momentum frame, express the total squared four-momentum of the neutron plus pion at the threshold for their production, in terms of the neutron mass m_n, the π meson mass m_π, and the speed of light c. ***(2 marks)***

(c) Working in the lab frame, express the total squared four-momentum of the proton plus gamma ray in terms of the energy E_p of the incoming proton and the energy E_γ of the incoming gamma. Make the assumptions that $E_p >> m_p c^2$, and that the particles collide head-on. ***(3 marks)***

(d) Express the minimum energy E_p of the proton necessary for the reaction to proceed in terms of E_γ, m_n, m_π and c, assuming that $m_n = m_p$. Evaluate this quantity giving your answer in electron volts. ***(4 marks)***

4 The interval between closely separated events in the vicinity of a non-rotating black hole can be derived from the Schwarzschild metric and is given by

$$ds^2 = -\left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 + \frac{dr^2}{(1 - \frac{2GM}{rc^2})} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2,$$

where M is the mass of the black hole, (ct, r, θ, ϕ) are time plus three spatial coordinates in the spherical polar coordinate system, G is Newton's gravitational constant, and c is the speed of light.

(a) In Eddington-Finkelstein coordinates a new coordinate v replaces the t coordinate, where v is related to t and r by

$$ct = v - r - \frac{2MG}{c^2} \ln \left| \frac{rc^2}{2MG} - 1 \right|.$$

Show that in this coordinate system, the interval ds^2 becomes

$$ds^2 = -\left(1 - \frac{2GM}{rc^2}\right) dv^2 + 2dvdr + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2.$$

(4 marks)

(b) Consider photons on radial trajectories. Show that these trajectories are described by

$$dv^2 \left(1 - \frac{2GM}{rc^2}\right) = 2dvdr.$$

(1 mark)

(c) One solution of this equation is $v = v_0$, where v_0 is a constant. By considering the effect of this constraint on v as defined in (a) above, show that this solution represents photons whose distance from the origin is decreasing with time on either side of the Schwarzschild radius $2GM = rc^2$. ***(2 marks)***

(d) A second solution is

$$2dr = \left(1 - \frac{2GM}{rc^2}\right) dv.$$

Integrate this differential equation to show that photons on these trajectories obey

$$v - 2\left(r + \frac{2GM}{c^2} \ln \left| \frac{rc^2}{2GM} - 1 \right| \right) = A,$$

where A is a constant. ***(3 marks)***

5 The metric for trajectories along radial lines passing through observers on Earth in comoving coordinates is

$$ds^2 = -c^2 dt^2 + a(t)^2 dr^2.$$

At time t_0, now, a galaxy comoving with the expansion that is not currently observable here on Earth is receding from us at a recession velocity of v_p.

(a) In terms of v_p, the Hubble constant H_0 and the present value of the scale factor $a(t_0)$, what are the current proper distance d_p and comoving distance r to the object? ***(2 marks)***

(b) Consider a Universe where the time rate of change of the scale factor is a constant. Show that the value of the scale factor at time t is given by

$$a(t) = a(t_0)(1 + H_0(t - t_0)).$$

(3 marks)

(c) The source emits a photon directed towards us. Show that the time τ for the photon to travel here from the source is given by

$$\tau = \frac{1}{H_0}\left(e^{\frac{v_p}{c}} - 1\right).$$

(3 marks)

(d) Show that when d_p is much less than the Hubble length, $\tau \simeq d_p/c$. ***(1 mark)***

(e) What is the redshift of the photon when it arrives at Earth in terms of v_p and c? ***(1 mark)***

End of Question Paper

Equations for PHY314 examination, 2006-2007

In the following, primed coordinates are moving to the right along the *x*-axis with respect to unprimed coordinates, and τ represents the time measured on a clock at rest with respect to and located next to an observer. In the 'Addition of velocities' formula, *u* is the velocity with which observer A is moving to the right with respect to observer B, and v_A and v_B are the velocities of the same projectile measured by observers A and B. In the Doppler shift and cosmological formulae, the subscript *e* denotes the quantity measured at rest with respect to the emitting source, and the subscript *o* refers to a quantity measured at reception by a remote, co-moving observer.

<u>Lorentz transformations</u>: $$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \quad \text{where} \quad \beta = \frac{v}{c}, \ \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

<u>Time dilation</u>: $\Delta t' = \gamma \Delta \tau$

<u>Lorentz contraction</u>: $L' = \dfrac{L}{\gamma}$

<u>Addition of velocities</u>: $$v_B = \frac{v_A + u}{1 + \dfrac{u v_A}{c^2}}$$

<u>Doppler shift</u>: $$1 + z = 1 + \frac{\Delta\lambda}{\lambda_e} = \sqrt{\frac{c+v}{c-v}}$$

<u>Red shift due to the expansion of the Universe</u>: $$1 + z = \frac{a(t_0)}{a(t_e)}$$

<u>Hubble's law</u>: $$\frac{H_0 d_L}{c} = z + \frac{1}{2}(1 - q_0) z^2$$ where $H_0 = 72$ km s^{-1} Mpc^{-1} and $q_0 = -0.2$