MATHEMATICAL PHYSICS

Attempt questions 1 and 4, and one other question from EACH of sections A and B (making a total of four questions).

SECTION A

1. **Compulsory**

(a) Given the matrices

$$A = \begin{pmatrix} -2 & 2 \\ 2 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 4 & 2 & 0 \\ 2 & 3 & -1 \\ 0 & -1 & 5 \end{pmatrix},$$

evaluate the determinant and the trace of each matrix. [2]

(b) Find the eigenvalues and normalised eigenvectors of the matrix A above. [3]

(c) The moment of inertia tensor for a set of particles, with positions $\boldsymbol{r}_\alpha$ and masses m_α is

$$I = \sum_\alpha m_\alpha[(\boldsymbol{r}_\alpha . \boldsymbol{r}_\alpha)\,\mathbb{1} - (\boldsymbol{r}_\alpha \otimes \boldsymbol{r}_\alpha)].$$

A rigid body consists of three equal masses m, which are located at positions $\boldsymbol{r}_1 = (a, -a, a)$, $\boldsymbol{r}_2 = (a, 2a, 0)$ and $\boldsymbol{r}_3 = (2a, a, -2a)$, joined by a light frame. Calculate the moment of inertia tensor I. [3]

(d) Show that the matrices $CC^\dagger$ and $C + C^\dagger$ are Hermitian, for *any* matrix C, not necessarily Hermitian. [2]

2. (a) Show that an orthogonal matrix must have:

i. Determinant ± 1. [1]

ii. Eigenvalues with unit modulus, $|\lambda| = 1$. [1]

(b) Rotations in two dimensions are given by the orthogonal matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Determine the eigenvalues and eigenvectors of this matrix. [3]

(c) Rotations in three dimensions are given by 3×3 orthogonal matrices with determinant $+1$. The secular equation is thus a cubic with real coefficients. Use this, and the results of (a), to show that the eigenvalues of the matrix must be 1, $e^{i\phi}$ and $e^{-i\phi}$, where ϕ is some real constant. [3]

(d) Explain, with reasons, the physical significance of:

i. The eigenvector corresponding to the eigenvalue 1 in part (c). [1]

ii. The constant ϕ. [1]

3. In quantum mechanics, the time dependence of a state is given by a unitary operator $\hat{U}(t)$, defined so that

$$\psi(x,t) = \hat{U}(t)\,\psi(x,0).$$

$\psi(x,t)$ represents the state of the system at time t, which satisfies the time dependent Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = \hat{H}\,\psi(x,t).$$

(a) Show that, if the Hamiltonian $\hat{H}$ is independent of t, $\hat{U}$ can be expressed as [3]

$$\hat{U}(t) = \exp\left(-i\hat{H}t/\hbar\right).$$

(b) Show also that $\hat{U}$ in (a) is unitary, and that this implies that the time evolution maintains the normalisation of the wavefunction. [2]

(c) The Hamiltonian of a particular two-state system is represented by the matrix

$$H = \frac{\hbar\omega}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Show that $H^2 = (\hbar\omega)^2\mathbb{1}$, and hence, or otherwise, demonstrate that [3]

$$\hat{U}(t) = \begin{pmatrix} \cos\omega t - \frac{i}{\sqrt{2}}\sin\omega t & -\frac{i}{\sqrt{2}}\sin\omega t \\ -\frac{i}{\sqrt{2}}\sin\omega t & \cos\omega t + \frac{i}{\sqrt{2}}\sin\omega t \end{pmatrix}.$$

(d) The system is initially in the state $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Calculate, as a function of time, the expectation value of the operator represented by the matrix [2]

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(The matrix $\exp(M)$ is defined by the power series

$$\exp(M) = \mathbb{1} + M + \frac{M^2}{2!} + \frac{M^3}{3!} + \cdots + \frac{M^n}{n!} + \cdots$$

and if M, N commute, the relationship $\exp(M+N) = \exp(M)\exp(N)$ holds.)

SECTION B

4. **Compulsory**

(a) State the Cauchy-Riemann conditions and determine which of the following functions is analytic throughout the complex plane: [2]

(i) z^2, (ii) $z + z^*$, (iii) iz, (iv) $\log z$.

(b) Evaluate the residues of the following functions at the points indicated: [3]

(i) $\dfrac{1}{2+3z}$ at $z = -\dfrac{2}{3}$ (ii) $\dfrac{1}{z(1-z)^2}$ at $z = 1$ (iii) $\dfrac{1}{\cos z + 1}$ at $z = \pi$.

(c) Develop the first three non-zero terms of the Laurent expansions, about the origin, for

$$f(z) = \frac{1}{z(z^2 - 1)}$$

which are valid (i) when $|z| < 1$ and (ii) when $|z| > 1$. [2]

(d) Consider the conformal transformation

$$w = u + iv = z + z^{-1},$$

where $z = x + iy$. Obtain expressions for u and v in terms of x and y. What curves do the circles $x^2 + y^2 = a^2$ map on to? Sketch these curves for $a = \frac{1}{2}$, 1, and 2. [3]

5. (a) Explain, with an example, the meanings of the terms *multiple-valued function*, *branch cut* and *Riemann surface*. [3]

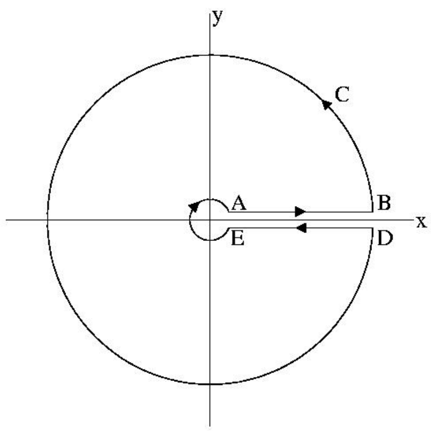
(b) Use the residue theorem to show that

$$I = \int_0^\infty dx \, \frac{x^a}{(1+x^2)^2} = \frac{\pi}{4}(1-a)\sec(\pi a/2),$$

where $-1 < a < +3$. [6]

(c) What happens to the integral when $a > 3$? [1]

Hint: One method of evaluating the integral is to use the contour shown below:



6. (a) Show that both the real and imaginary parts of an analytic complex function $f(z) = f(x+iy)$ satisfy Laplace's equation in two dimensions, [2]

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.$$

(b) Consider the function $f(z) = (2/\pi)\log z$. Calculate the imaginary part of this function, and show that it satisfies the boundary conditions $\phi(x,y) = 0$ on the line $y = 0$, and $\phi(x,y) = 1$ on the line $x = 0$, $y > 0$. [3]

(c) For the conformal mapping

$$z = i\,\frac{1-w}{1+w}, \quad w = u + iv,$$

find x and y as functions of u and v. Show that the line $y = 0$ maps onto the unit circle $u^2 + v^2 = 1$, and the line $x = 0$, $y > 0$ maps onto the line $v = 0$ between $u = -1$ and $u = +1$. [3]

(d) Use the results of (b) and (c) to obtain the electrostatic potential $\phi(u,v)$ in the space between the half circle $u^2 + v^2 = 1$, $v > 0$ and the line $v = 0$, when $\phi = 0$ on the semi-circular boundary and $\phi = 1$ on the line segment $-1 \le u \le 1$. [2]

END OF QUESTION PAPER