Further Quantum Mechanics

Attempt three of the five questions on the paper.

1. (a) Explain how the variational method is used to find the ground state energy, E_1, associated with a given Hamiltonian $\hat{H}$. [1]

   (b) Show that

$$\frac{\langle\psi|\hat{H}|\psi\rangle}{\langle\psi|\psi\rangle} \geq E_1,$$

   where ψ is any trial wavefunction. [3]

   (c) Consider the trial wavefunction $\psi(x) = e^{-\beta|x|}$, where β is a variational parameter. Sketch this wavefunction, and its first and second derivatives, paying particular attention to the region around $x = 0$. [2]

   (d) Use this wavefunction to estimate the ground state energy of the harmonic oscillator, with [4]

$$\hat{H} = -\frac{\hbar^2}{2m_e}\frac{d^2}{dx^2} + \frac{1}{2}m_e\omega_c^2 x^2.$$

2. The Pauli matrices are

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

   (a) Show that $[\sigma_x, \sigma_y] = 2i\sigma_z$, and that $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \mathbb{1}$. [2]

   (b) Find the eigenvalues and eigenvectors of each of σ_z and σ_y. [3]

   (c) An electron is placed in a region where there is a uniform magnetic field, magnitude B, in the y direction. What is the Hamiltonian for the spin part of the wavefunction for the electron? [1]

   (d) If the electron is initially in the state $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, calculate the state vector as a function of time. What are the possible outcomes of a measurement of σ_y, and with what probability will each occur if the measurement is made at time t? [4]

3. (a) How does the vector potential $\boldsymbol{A}$ in electromagnetism relate to the magnetic field $\boldsymbol{B}$? Explain what is meant by *gauge invariance* in the context of electromagnetism. [3]

(b) Give a suitable form for $\boldsymbol{A}$ which corresponds to a uniform magnetic field of magnitude B acting in the z direction. [1]

(c) A hydrogen atom is placed in a uniform magnetic field. Use first order perturbation theory to calculate the energy shift of the ground state due to the quadratic term in the Hamiltonian

$$\hat{H}' = \frac{e^2 A^2}{2m_e},$$

where e and m_e are the charge and mass of the electron. [3]

(The normalised ground state wavefunction of the hydrogen atom is $\phi(\boldsymbol{r}) = (\pi a_0^3)^{-\frac{1}{2}} \exp(-r/a_0)$, where $a_0 = 0.0529$ nm is the Bohr radius.)

(d) Find a numerical value for the field B at which the energy shift is equal to the Zeeman splitting $g\mu_B B$. You may assume that $g = 2$. [2]

(e) Sketch what happens to the zero-field lowest energy level of the atom, as a function of magnetic field. [1]

4. For a system of N identical particles, in a state $\psi(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots \boldsymbol{r}_N)$, the probability of finding any one particle in a volume element $d^3\boldsymbol{r}$ about the point $\boldsymbol{r}$ is $P_1(\boldsymbol{r})\, d^3\boldsymbol{r}$, where

$$P_1(\boldsymbol{r}) = \int d^3\boldsymbol{r}_2 \int d^3\boldsymbol{r}_3 \ldots \int d^3\boldsymbol{r}_N\, |\psi(\boldsymbol{r}, \boldsymbol{r}_2, \ldots \boldsymbol{r}_N)|^2$$

(a) Write down a suitable wavefunction for two, non-interacting, indistinguishable fermions in the orthonormal states $\phi_1(\boldsymbol{r})$ and $\phi_2(\boldsymbol{r})$. Give the equivalent wavefunction for bosons. [2]

(b) In both cases, show that $P_1(\boldsymbol{r}) = \frac{1}{2}\left[|\phi_1(\boldsymbol{r})|^2 + |\phi_2(\boldsymbol{r})|^2\right]$. [2]

Two indistinguishable, non-interacting, spin-$\frac{1}{2}$ particles move in a one-dimensional infinite well of width a (that is, $V(x) = 0$ for $|x| \leq a/2$, $V(x) = \infty$ for $|x| > a/2$).

(c) What are the energies and wave-functions for the lowest energy configurations with (i) total spin $S = 0$, (ii) $S = 1$? [2]

(d) A weak interaction between the particles is switched on, with a short range potential of the form $V_0\, \delta(x_1 - x_2)$. Starting from the infinite potential well wavefunctions, use first order perturbation theory to calculate the energy shifts for the two spin configurations. [4]

5. If the Hamiltonian of a time dependent system is split into two parts with the form $\hat{H}(x,t) = \hat{H}_0(x) + \hat{H}'(x,t)$, the wavefunction can be written as

$$\psi(x,t) = \sum_i c_i(t)\,\phi_i(x)\,e^{-iE_i t/\hbar}$$

where ϕ_i and E_i are the eigenfunctions and eigenvalues of $\hat{H}_0$.

(a) Derive the equations for the time dependence of $c_i(t)$. [3]

(b) Show that, if the system is initially in state ϕ_n at $t = 0$, the first order approximation to $c_m(t)$ $(m \neq n)$ is

$$c_m(t) = \frac{1}{i\hbar}\int_0^t dt'\, H'_{mn}(t')\, e^{i(E_m - E_n)t'/\hbar},$$

where $H'_{mn}(t) = \langle m|\hat{H}'|n\rangle$. [2]

(c) A harmonic oscillator is in its ground state at time $t = 0$. For $t > 0$, it is subject to a time varying potential

$$V(x,t) = F_0 x e^{-t/\tau},$$

where F_0 is a constant and τ the decay time of the potential. Use first-order time-dependent perturbation theory to calculate the probability of finding the oscillator in the first excited state at time t. What is the behaviour for $t \to \infty$? [4]

(d) Is there any possibility of finding the oscillator in a higher excited state? [1]

You may assume that the matrix elements of x between harmonic oscillator states n and n' satisfy

$$\langle n'|x|n\rangle = \sqrt{\frac{\hbar}{2m_e\omega_c}}\left(\sqrt{n}\,\delta_{n',n-1} + \sqrt{n+1}\,\delta_{n',n+1}\right).$$

END OF QUESTION PAPER