The University Of Sheffield.

DEPARTMENT OF PHYSICS & ASTRONOMY

Autumn 2006-2007

MATHEMATICAL METHODS FOR PHYSICS & ASTRONOMY 2 HOURS

Answer question ONE (COMPULSORY) and TWO others.

A formula sheet and table of physical constants is attached to this paper.

All questions are marked out of ten. The breakdown on the right-hand side of the paper is meant as a guide to the marks that can be obtained from each part.

TURN OVER

1. COMPULSORY

(a) Plot $z = \frac{1}{2} + \frac{\sqrt{3}}{2} i$ on an Argand diagram.

Express z in the form $re^{i\phi}$.

Evaluate z^3. [2]

(b) Evaluate the following integrals;

$$\int_0^{2\pi} d\theta \sin^2 4\theta \qquad\qquad \int_0^{L} dx \cos^2 \frac{n\pi x}{L}.$$ [2]

(c) Find the value of ω such that the function

$$\Psi(x, y, z, t) = A \exp(i\omega t - ik_x x - ik_y y - ik_z z) = Ae^{i(\omega t - \mathbf{k.r})}$$

is a solution of the wave equation in three dimensions:

$$\nabla^2 \Psi(x, y, z, t) = \frac{1}{c^2} \frac{\partial^2 \Psi(x, y, z, t)}{\partial t^2}.$$ [3]

(d) Find possible values for the constant A such that the solution to the following equation satisfies $y(0) = y(L) = 0$.

$$\frac{d^2 y}{dx^2} = Ay.$$ [3]

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2. (a) Explain why a periodic function $f(\theta)$ that satisfies $f(\theta) = f(\theta + 2\pi)$ for all θ may be written in a *Fourier Series.* [2]

(b) Explain what simplifications can arise if the function $f(\theta)$ is either an even or an odd function. [2]

(c) Such a periodic function $f(\theta)$ is defined over the range $-\pi < \theta < \pi$ as follows:

$$f(\theta) = \begin{cases} 0 & -\pi < \theta < -\pi/4 \\ h & -\pi/4 < \theta < \pi/4 \\ 0 & \pi/4 < \theta < \pi \end{cases}$$

Sketch the function in the range $-2\pi < \theta < 2\pi$. [2]

(d) Evaluate the Fourier series for the function given in (c) up to and including terms with $n = 4$. [4]

TURN OVER

3. The Fourier transform $F(k)$ of a function $f(x)$ is defined as follows:

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}x \; f(x) e^{ikx} \; .$$

(a) Evaluate the Fourier Transforms of the following functions:

(i) $f_a(x) = 1 \qquad |x| < d$ [2]

$\qquad = 0 \qquad$ otherwise

(ii) $f_b(x) = e^{-\alpha|x|}; \qquad \alpha > 0.$ [2]

(b) In both cases (i) and (ii) sketch both the function $f(x)$ and its Fourier transform, $F(k)$. [1]

(c) For $F_b(k)$ (the Fourier Transform of $f_b(x)$ above), find the values of k such that $F_b(k) = \frac{1}{2} F_b(0)$. [1]

(d) A three dimensional Fourier transform of a spherical function is defined as follows:

$$\frac{1}{(2\pi)^{3/2}} \int_0^{2\pi} \mathrm{d}\phi \int_0^{\infty} \mathrm{d}r \; r^2 f(r) \int_0^{\pi} \mathrm{d}\theta \sin\theta \, e^{ikr\cos\theta}$$

Evaluate this integral for a spherical shell: $f(r) = A\delta(r - a)$. [3]

(e) Give an example where Fourier techniques are used in either Physics or Astronomy. [1]

CONTINUED

4. (a) Solve the following differential equations where k and α are real:

(i) $\dfrac{d^2y(x)}{dx^2} = -k^2 y(x), \qquad y(0) = a, \qquad \left.\dfrac{dy}{dx}\right|_{x=0} = 0;$ [2]

(ii) $\dfrac{d^2y(x)}{dx^2} = \alpha^2 y(x), \qquad y(0) = b, \qquad \left.\dfrac{dy}{dx}\right|_{x=0} = 0.$ [2]

Explain why the solutions are physically different. Give examples of situations where each of these equations might be used. [1]

(b) Laplace's equation in two dimensions is given by

$$\frac{\partial^2 V(x,y)}{\partial x^2} + \frac{\partial^2 V(x,y)}{\partial y^2} = 0 .$$

In a certain problem it is known that

$$V(x,0) = A\cos kx \text{ for all } x.$$

Explain why it is appropriate to solve this equation by separation of the variables. [1]

Find a solution that satisfies $\lim_{y\to\infty} V(x,y) \to 0.$ [4]

TURN OVER

5. The variation of temperature in a uniform medium satisfies the diffusion equation (h^2 is the thermal diffusivity).

$$h^2 \frac{\partial^2 T(x,t)}{\partial x^2} = \frac{\partial T(x,t)}{\partial t}.$$

At $x = 0$, $T(0,t) = T_0 + T_1 \cos \omega t$, where $T_1 < T$.

(a) Explain why it is useful to solve for a *complex* function, $\Theta(x, t)$, such that $\mathrm{Re}\,\Theta(x,t) = T(x,t)$. [2]

(b) Show that it is possible to solve for $\Theta(x, t)$ for $x > 0$, using separation of the variables to obtain

$$\Theta(x, t) = T_0 + A \exp\left(i\omega t + \frac{ix}{\lambda} + \frac{x}{\lambda} \right) + B \exp\left(i\omega t - \frac{ix}{\lambda} - \frac{x}{\lambda} \right).$$ [3]

Give an expression for λ. [1]

(c) Use boundary conditions and physical reasoning to obtain A and B. [1]

(e) Calculate the amplitude and phase of the temperature variation on the surface of a slab of thickness 1 cm when the temperature of the opposite face is cycled at 1 Hz with an amplitude of 10 K. The diffusivity of the slab is $h^2 = 10^{-4}\,\mathrm{m^2\,s^{-1}}$. [3]

END OF EXAMINATION PAPER