The University Of Sheffield.

DEPARTMENT OF PHYSICS AND ASTRONOMY

Data Provided:
Linear-Linear Graph Paper

Autumn 2006-2007

TOPICS IN CLASSICAL PHYSICS **2 HOURS**

Answer question ONE (COMPULSORY) and TWO other questions.

A formula sheet and table of physical constants is attached to this paper.

All questions are marked out of ten. The breakdown on the right-hand side of the paper is meant as a guide to the marks that can be obtained from each part.

1. COMPULSORY

a) A ship is partially submerged in water, and will experience 'wave drag' due to the generation of surface waves as it moves. In deep water, wave drag depends on the length of the vessel l, the velocity of the vessel v, and the acceleration due to gravity g.

Combine l, v and g into a dimensionless group. [2]

b) An oscillator undergoes 42 oscillations until its amplitude has decayed from an initial 1 cm to 0.3679 cm. Calculate the Quality, Q, of the oscillator. [2]

c) Long water waves are characterised by a dispersion relation,

$$\omega(k) = (gk)^{1/2}.$$

Calculate the phase velocity and the group velocity of water waves of 5 m wavelength, which can be considered 'long'. [2]

d) An athlete wishes to exploit the fictitious centrifugal and Coriolis forces for a long jump world record attempt. Which location on Earth should the athlete choose to maximise the helpful effects of these fictitious forces?

Does the direction of run-up matter? If so, which direction should the athlete choose? [2]

e) A point mass moving in a plane is described by two generalised coordinates, its distance r from an arbitrary origin, and its angle ϕ with respect to an arbitrary reference axis that is defined to have $\phi = 0$.

What are the two Lagrangian generalised velocities of this mechanical system?

What are their SI units (or dimensions)?

Why is it incorrect to simply add the squares of the two generalised velocities to calculate the square of the conventional (Cartesian) velocity if you need it, say, to calculate the kinetic energy of the point mass? [2]

2.

The motion of a simple pendulum is described by the differential equation

$$\ddot{\varphi} + \frac{g}{l}\sin\varphi = 0,$$

where φ is the angle the pendulum makes to the vertical, l the length of the string, and g the acceleration due to gravity. The amplitude of the swing is given by the maximum angle, φ_{MAX}.

For small amplitudes φ_{MAX}, this differential equation can be simplified by the approximation $\sin\varphi \approx \varphi$. The simplified equation describes sinusoidal oscillations with angular frequency $\omega = (g/l)^{1/2}$ and period $\tau = 2\pi(l/g)^{1/2}$.

For higher amplitudes, this is no longer valid and the period τ must be written in the form

$$\tau = 2\pi\sqrt{\frac{l}{g}}\, f(\varphi_{MAX}),$$

where f is an unknown function of the amplitude.

a) What are the dimensions of $f(\varphi_{MAX})$? [1]

b) What is the value of f as $\varphi_{MAX} \to 0$, $\lim_{\varphi_{MAX}\to 0} f(\varphi_{MAX})$? [1]

c) Assuming that f can be expanded into a power series,

$$f(\varphi_{MAX}) = a_0 + \sum_{n=1}^{\infty} a_n \varphi_{MAX}^n,$$

show that $a_0 = 1$, and demonstrate that all coefficients a_n with an odd index (a_1, a_3, a_5, …) must be equal to zero. [3]

d) Steps a) to c) provide an approximate form for f,

$$f(\varphi_{MAX}) \cong 1 + a_2\varphi_{MAX}^2,$$

by truncating the power series after the first non- zero coefficient.

The table below shows the measured period τ of a pendulum with length $l = 1$ m for different amplitudes φ_{MAX}. Assuming the approximate form for f, transform the tabulated data in such a way that they can be plotted in a graph, where both axis are dimensionless and transformed data follows a straight line with zero intercept and a slope related to a_2.

Plot a graph to show that the transformed data follows a good straight line for small φ_{MAX}, but curves away upwards from the straight line for larger values of φ_{MAX}. Explain why this happens.

What is the best way to find a_2, by fitting a straight line through ***all*** the data, or by ***excluding the data for large φ_{MAX}*** to get a better fit to the remaining data? Carry out the appropriate fit, and extract a_2 from it. [5]

φ_{MAX} (degrees)	5	10	15	20	25	30	40	50	60	70	80
τ (s)	2.007	2.010	2.015	2.021	2.030	2.041	2.069	2.106	2.152	2.209	2.278

TURN OVER

3.

There are two types of waves on the surface of deep water, depending on their wavelength compared to a characteristic wavelength, λ_{char}.

For long waves ($\lambda >> \lambda_{\text{char}}$), surface tension can be neglected, whereas for short waves ($\lambda << \lambda_{\text{char}}$), also called 'capillary waves', surface tension makes an important contribution to the wave's energy. The two different types of wave have phase velocities c_{long} and c_{short}, given as a function of wavelength by

$$c_{long} = \sqrt{\frac{g\lambda}{2\pi}}$$
$$c_{short} = \sqrt{\frac{2\pi\sigma}{\rho\lambda}}.$$

Here λ is the wavelength, ρ is the density of water, 10^3 kg m^{-3}, and σ is a constant that quantifies the strength of surface tension and for water has the value $\sigma = 0.07$ N m^{-1}.

Note that for $\lambda \approx \lambda_{\text{char}}$, neither of the above equations applies precisely, but there will be a smooth transition between the two regimes.

a) Give the dispersion relations for long and short water waves in the form $\omega(k)$. [2]

b) Sketch a graph of $\omega(k)$ including both the long and short wave regimes.

In drawing the graph, sketch the $\omega(k)$ relation in both the short and long wavelength regimes as a ***solid line*** and annotate those lines with the power of k that $\omega(k)$ follows.

Link the two regimes with a ***dashed line*** across the transition regime in which neither the long nor the short wave $\omega(k)$ is precisely correct.

In your graph, numerical values are not required on either axis, but clearly annotate the k axis to show which sections correspond to the long and short wavelength regimes and which to the transition region. [4]

c) Estimate the characteristic wavelength λ_{char} that marks the transition between the short and long wavelength regime. [2]

d) Identify, either from inspection of your sketch or by some other argument, a region where there is no dispersion of the water waves, that is where $\omega \approx ck$ with a k-independent phase velocity c. Estimate the phase velocity c in this dispersion-free region. [2]

4.

A suspension of spherical particles in water is held in a test tube. The radius of the particles is $r = 100$ nm; their density is $\rho = 5 \times 10^3$ kg m^{-3} and the density of water is $\rho = 10^3$ kg m^{-3}.

The particles settle to the bottom of the test tube under the influence of gravity with a velocity v_{grav} which is controlled by the balance of gravity and viscous drag.
The Stokes viscous drag force is given by

$$F_{\mathrm{drag}} = 6\ \pi\eta r v,$$

where the viscosity of water $\eta = 10^{-3}$ N s m^{-2}.

a) Calculate the velocity v_{grav} with which the particles settle under gravity. How long will they take to settle over a distance of 2 cm? [2]

b) To shorten the time required to settle, the test tube is mounted in a centrifuge that allows it to spin around a central hub on arms of $R = 20$ cm.

How many revolutions per minute (rpm) of the centrifuge will generate a centrifugal acceleration that is 1000 times larger than acceleration due to gravity (*i.e.* 1000g)? With what velocity v_{Cf} will the particles now settle? [4]

c) When settling with this new v_{Cf} under 1000g spin, the particles will also experience a Coriolis force and will move with a corresponding velocity, given again by Stokes' drag equation.

If the axis of rotation is defined as the polar axis and the direction of centrifugal force as 'radial', in which direction will the Coriolis force act, assuming the rotation is counter-clockwise as viewed from above? You may give the answer in the form of a sketch, or in words.

Calculate the ratio of velocity due to the Coriolis force (v_{Cor}) and velocity due to centrifugal force, $v_{\mathrm{Cor}}/v_{\mathrm{Cf}}$. Does this result suggest that there may be problems with centrifuging because the particles may be forced to the sidewalls of the test tube before they reach the bottom? [4]

TURN OVER

5.

A uniform, flexible rope, of length D and mass M, hangs over the edge of a table which has a frictionless surface. (The height of the table is greater than D). The length of the rope which hangs over the edge at any time t is given by $l(t)$, where $l(t) < \mathrm{D}$: the remainder of the rope lies on (or rather, slides across) the surface of the table.

This mechanical system can be entirely characterised by the single dimensionless generalised coordinate

$$f(t) = l(t)/D,$$

which is the fraction of rope dangling over the edge at a given time.

In the calculations below consider only times such that $l(t) \leq D$, that is that $f(t) \leq 1$.

a) Make a sketch of the physical situation described above. [1]

b) Calculate the kinetic energy T, of the rope, as a function of $\mathrm{d}f/\mathrm{d}t$. [2]

c) Calculate the potential energy V, of the rope as a function of f.

Normalise V so that it is zero for $f = 0$, when the rope lies entirely on the table surface ($l(t) = 0$). [2]

d) Combine T and V into the Lagrangian for this system and set up the Lagrangian equation of motion. [2]

e) Give the general solution $f(t)$ to the Lagrangian equation of motion.

[Note that the Lagrangian equation of motion will have two linearly independent solutions that need to be combined into the complete solution.]

f) Give the special solution which corresponds to the conditions when $f(0) = f_0$ describes the fraction of rope that initially hangs over the edge of the table and $\mathrm{d}f/\mathrm{d}t = 0$ at $t = 0$ (here $0 < f_0 < 1$). [3]

END OF QUESTION PAPER