The University Of Sheffield.

DEPARTMENT OF PHYSICS AND ASTRONOMY

Autumn 2006-2007

INTRODUCTORY MATHEMATICS FOR PHYSICISTS AND ASTRONOMERS **3 HOURS**

Attempt ALL questions.

The breakdown on the right-hand side of the paper is meant as a guide to the marks that can be obtained from each part.

There are 200 possible marks for this paper.

TURN OVER

Unit 1.

1. Simplify the following expression to the form x^a:

$$\frac{\left(\sqrt{x}\right)^{\frac{2}{3}} x^2}{x^{\frac{4}{3}}}.$$ [2]

2. Combine the following expression into a single term:

$$4 \ln x + 2 \ln y + \ln z.$$ [2]

3. Separate the following expression into two partial fractions:

$$\frac{3x}{(x+2)(x+3)}.$$ [4]

4. Combine the following into a single fraction:

$$\frac{2}{x^2} + \frac{3}{x}.$$ [2]

Unit 2.

5. Find all the solutions of the equation $2\cos^2 x - \cos x - 1 = 0$, for $0 \le x \le \pi$. [6]

6. A certain amount of the radioactive isotope of thorium ^{232}Th was produced during a supernova explosion 2 billion years ago. This isotope decays according to the exponential law $N(t) = N_0 e^{-t/t_0}$, where N_0 and N are the initial number of atoms and the number of atoms after time t, respectively, and $t_0 = 2\times10^{10}$ years. Calculate the fraction of initial atoms that have not decayed since the explosion.
What time is needed for one half of the initial atoms of thorium to decay? [6]

7. Find the first derivatives $\frac{\mathrm{d}y}{\mathrm{d}x}$ of the functions $y = f(x)$:

a) $y = (3 - 2x^5)^4$ [6]

b) $y = \frac{\sin(\omega x + \phi)}{x}$ [6]

8. Show that the time derivative $\frac{\mathrm{d}I}{\mathrm{d}t}$ of the function $I = ae^{-\lambda t}\cos(bt)$ is

$$\frac{\mathrm{d}I}{\mathrm{d}t} = -ae^{-\lambda t}\sqrt{\lambda^2 + b^2}\cos\left(bt + \arccos\left(\frac{\lambda}{\sqrt{\lambda^2 + b^2}}\right)\right).$$

Use the relation

$$\alpha\cos x + \beta\sin x = \sqrt{\alpha^2 + \beta^2}\cos\left(x + \arccos\left(\frac{\alpha}{\sqrt{\alpha^2 + \beta^2}}\right)\right).$$ [8]

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Unit 3.

9. If $z = 1 - i$, calculate and plot on an Argand diagram the following complex numbers:

a) z^*,
b) $3z$,
c) $2iz$,
d) z^2,
e) $2/z$. [5]

10. Find the modulus, argument and complex conjugate of the following complex numbers:

a) $1+2i$,
b) $-3+2i$,
c) $6e^{i\pi/5}$ [4.5]

11. If $z_1 = 2 + 4i$ and $z_2 = -2 + i$ calculate:
a) z_1+z_2,
b) z_1z_2,
c) $\frac{z_1}{z_2}$. [4.5]

12. Find the two square roots of $1 - i$. [4]

13. An oscillating voltage of amplitude $|V|$ and angular frequency ω is applied to a circuit consisting of a capacitor and resistor connected in parallel. Show that the magnitude of the current which flows through the circuit is given by

$$\frac{|V|\sqrt{1+(\omega RC)^2}}{R}.$$

State the formula which gives the phase difference between the current and voltage. What is this phase difference for $\omega = 0$ and $\omega \to \infty$? [6]

[Hint: the complex reactance of a capacitor is $\frac{1}{i\omega C}$.]

TURN OVER

Unit 4.

14. Evaluate the following integrals:

(i) $\int_0^\infty e^{-2x}\,\mathrm{d}x$ [3]

(ii) $\int_0^1 \frac{1}{x^{1/3}}\,\mathrm{d}x$ [3]

(iii) $\int_1^3 (5x-4)^2\,\mathrm{d}x$ [4]

15. Find the integrals:

(i) $\int 4\sin x\cos x\,\mathrm{d}x$ [4]

(ii) $\int xe^{x^2}\,\mathrm{d}x$ [4]

(iii) $\int \frac{x}{3x^2+2}\,\mathrm{d}x$ [4]

16. Using the method of partial fractions find

$$\int \frac{x}{(2x+1)(x+1)}\,\mathrm{d}x\,.$$ [6]

17. Show by integration that $\int \frac{1}{(a^2-x^2)^{1/2}}\,\mathrm{d}x \;=\; \sin^{-1}\left(\frac{x}{a}\right)+c$ where a and c are constants. [4]

18. Using the method of integration by parts, find

$$\int x^2 \sin x\,\mathrm{d}x\,.$$ [6]

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Unit 5.

19. Find the solution of the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = 2y + 1$ with the initial condition $y(x = 0) = 0$. [6]

20. Find the general solution of the differential equation $x\frac{\mathrm{d}y}{\mathrm{d}x} = y$. [8]

21. A spaceship of mass 10 tonnes moves towards Jupiter. The ship engines provide a constant accelerating force of 10^4 N. Write down Newton's Second Law as a second order differential equation of the motion of the spaceship assuming that gravitational forces can be neglected. Solve this equation to find the distance, x, as a function of time, t, with the initial conditions $x\ (t = 0) = 10^5$ km and speed $v\ (t = 0) = 10$ km/s. [8]

    After what time (in days) will the spaceship reach the orbit of Mars (x = 80 million kilometres)? What will the speed of the spaceship be at that time? Give answers to 2 decimal places. [6]

TURN OVER

Unit 6.

22. For the vectors $\mathbf{a} = 6\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 8\hat{\mathbf{k}}$, $\mathbf{b} = 3\hat{\mathbf{i}} - 5\hat{\mathbf{j}} + 7\hat{\mathbf{k}}$ and $\mathbf{c} = \hat{\mathbf{i}} - 8\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$ find:

(i) $\mathbf{3a} + 5\mathbf{b}$ [2]
(ii) $\mathbf{a.b}$ [2]
(iii) $\mathbf{a.(b \times c)}$ [2]

23. (a) Given the three points A (2, 2, 2), B (5, 4, 6) and C (0, −2,1), find the angle between $\overrightarrow{CA}$ (vector $\mathbf{a}$) and $\overrightarrow{CB}$ (vector $\mathbf{b}$). [3]

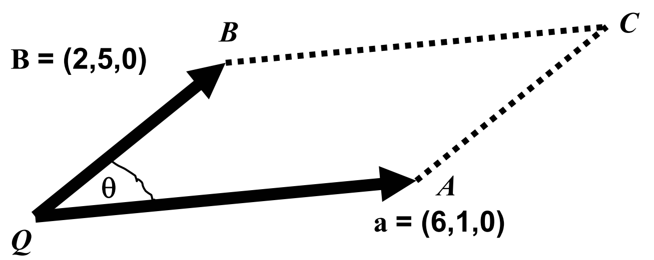
(b) Find the values of α, β and γ which make the vectors **a**, **b** and **c** mutually perpendicular, where:

$$\mathbf{a} = \alpha\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$$
$$\mathbf{b} = \hat{\mathbf{i}} + \beta\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$$
$$\mathbf{c} = 4\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + \gamma\hat{\mathbf{k}}$$

[4]

(c) Show that the planes $3x + 4y - 6z = 5$ and $2x + 3y + 3z = 0$ are perpendicular. [3]

(d) Let $\mathbf{a} = \overrightarrow{QA}$ and $\mathbf{b} = \overrightarrow{QB}$ be two vectors from Q (0, 0, 0) in the diagram below, representing two sides of a parallelogram. Find the area of the parallelogram $QACB$.



[3]

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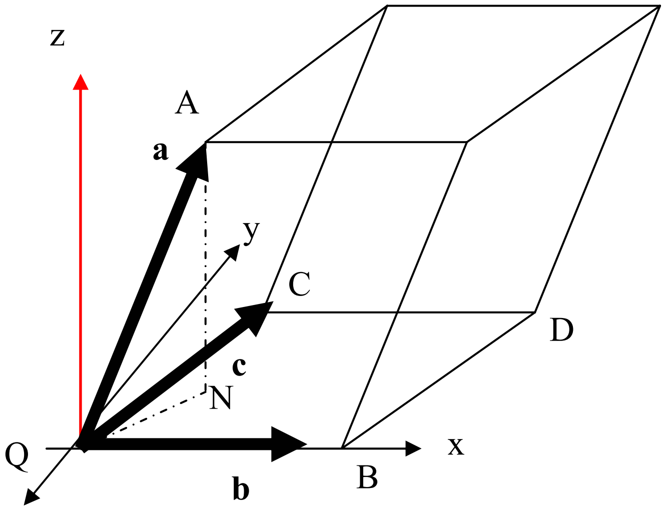
24. (a) The diagram below shows a parallelepiped in which the three adjacent sides originating from Q (0, 0, 0) are represented by the three vectors (units are centimetres):

$$\mathbf{a} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + 8\hat{\mathbf{k}}$$
$$\mathbf{b} = 9\hat{\mathbf{i}} + 0\hat{\mathbf{j}} + 0\hat{\mathbf{k}}$$
$$\mathbf{c} = 2\hat{\mathbf{i}} + 6\hat{\mathbf{j}} + \hat{\mathbf{k}}$$



Find the volume of the parallelepiped. [5]

(b) A force $\mathbf{F} = 14\hat{\mathbf{i}} - 4\hat{\mathbf{j}}$ newtons acts at the point P (1, 2, 0). Find its vector moment about the origin (0, 0, 0). [5]

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Unit 7.

25. Obtain $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at the point (6, 3) for the following functions:

(i) $8x + 4y - 3$; [2]

(ii) $\left(x^2 + y^2\right)^{1/2}$. [3]

26. Find the equation of the tangent plane at the point P (6, 7,–6) on the sphere $x^2 + y^2 + z^2 = 121$. Express your answer in the form $z = ax + by + c$ where a, b and c are constants. [6]

27. Find the stationary points of $f(x, y) = \frac{1}{3}x^3 - xy^2 - 2y$ and the value of $f(x, y)$ at these points. [3]

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Unit 8.

28. (a) What is the probability of drawing a King on four consecutive occasions from a standard pack of 52 playing cards if each selected card is **not returned** to the pack? [1]

(b) A coin is spun until a "heads" is obtained. What is the probability of spinning 6 "tails" before the first "heads"? [2]

(c) A box contains 8 red balls and 4 blue balls. What is the probability of drawing the first four balls as red and the fifth ball as blue if
(i) the balls are replaced after each draw and
(ii) the balls are not replaced? [3]

29. The continuous random variable x has a probability density function $f(x)$ given by

$$f(x) = bx(6-x) \quad 0 \le x \le 6$$

$$f(x) = 0 \quad \text{otherwise}$$

where b is a constant.

(a) Sketch the probability function. [2]
(b) Find the value of the constant b. [3]
(c) Find the value of $P(x > 3.3)$. [3]
(d) Find the mean. [2]
(e) Find the standard deviation, σ. [5]

30. Show that the function

$$p_n(x) = \frac{\lambda^n e^{-\lambda}}{n!} \quad (n = 0, 1, 2.....)$$

is a valid probability function. [4]

[Hint: Use the Taylor expansion for e^x.]

END OF QUESTION PAPER