The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Autumn Semester 2006–07

Numbers and Proofs — 2 hours

Answer ***Question 1*** *and three other questions. If you answer more than three of the questions 2 to 5, only your best three will be counted. Question 1 is worth* 28 marks*; the other questions are each worth* 20 marks.

1 (i) (a) Give the English names for the Greek letters ρ and ϕ. ***(1 mark)***

(b) Give the symbol for the set of integers. ***(1 mark)***

(ii) What does it mean for an integer a to divide another integer b? Prove that if $a|b$ and $a|c$, then $a|b+c$. Is it true that if $a|b$ *or* $a|c$, then $a|b+c$? ***(6 marks)***

(iii) Consider the statement:

If a natural number n is divisible by 24, then n^2 is divisible by 48.

(a) What is the converse of this statement? ***(1 mark)***

(b) What is the contrapositive of this statement? ***(1 mark)***

(c) Prove that this statement is true. ***(3 marks)***

(d) Is the converse true? Give a proof or a counterexample. ***(2 marks)***

(iv) State the *division algorithm* for polynomials in $\mathbb{R}[x]$. Use it to determine the remainder when $x^5 - x^3 + 2x^2 + 2x$ is divided by $x^2 - x - 2$. ***(6 marks)***

(v) State a simple condition in terms of f and its derivative f' for a number α to be a repeated root of a polynomial f. Use it to find a repeated root of the polynomial

$$x^4 + x^3 - 3x^2 - 5x - 2,$$

and hence determine all of its roots. ***(7 marks)***

2 (i) What is meant by the *highest common factor* (a, b) of integers a and b, not both zero? ***(2 marks)***

(ii) If a and b are two integers, and $a = qb + r$ for integers q and r, show that $(a, b) = (b, r)$. ***(5 marks)***

(iii) (a) A student buys some pens for 35p and some files for 91p. The total bill is £6.09. Use the Euclidean algorithm to determine how many of each she bought. ***(8 marks)***

(b) Another student bought some of the same pens and files, and was charged £3.29. Show that the cashier made an error. ***(5 marks)***

3 (i) One of the congruences $68x \equiv 118 \pmod{187}$ and $68x \equiv 119 \pmod{187}$ has no solutions. Which is it? Find all solutions to the other, expressing your answer in the form $x \equiv a \pmod{m}$. ***(7 marks)***

(ii) Find the general simultaneous solution to

$$x \equiv 46 \pmod{53} \qquad \text{and} \qquad x \equiv 1 \pmod{59},$$

expressing your answers in the form $x \equiv a \pmod{m}$. ***(7 marks)***

(iii) Find an integer which is five times a square and twice a cube. ***(3 marks)***

(iv) By writing the natural number a in the form $1000q + r$, for a suitable quotient q and remainder r, show that a is divisible by 8 if and only if the number formed by its final three digits is divisible by 8. ***(3 marks)***

4 (i) Let p be a prime number, and let a be a non-negative integer. Prove by induction that $a^p \equiv a \pmod{p}$, assuming any standard results on binomial coefficients that you need. ***(5 marks)***

(ii) What is $2007^{2007} \pmod{77}$? ***(6 marks)***

(iii) Recall that the *Fibonacci numbers* are defined by $F_1 = F_2 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 1$. Prove by induction that for all $n \geq 1$,

$$F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n+1}.$$ ***(4 marks)***

(iv) Prove by induction that $8^n + 6$ is divisible by 14 for all natural numbers n. ***(5 marks)***

5 (i) Prove that $\sqrt{20}$ is irrational. ***(3 marks)***

(ii) Give decimal expansions of both a rational number and an irrational number strictly between $0.\dot{1}23456789\dot{0}$ and $0.\dot{1}2345678\dot{9}$, explaining your answer briefly. ***(3 marks)***

(iii) Explain why the decimal expansion of any rational number m/n must eventually recur. ***(3 marks)***

(iv) Write the real number $0.2\dot{0}0\dot{7}$ as the ratio of two coprime natural numbers, expressing each in its *canonical prime factorisation*. ***(7 marks)***

(v) Write $x = 0.\dot{a}_1a_2a_3a_4a_5\dot{a}_6$, a recurring decimal with period 6. If x is written as a fraction in lowest terms, what is the largest denominator that x can have? What is the largest prime factor its denominator can have? ***(4 marks)***

End of Question Paper