SCHOOL OF MATHEMATICS AND STATISTICS

Autumn Semester 2005–06 **2 hours**

Numbers and Proofs

Answer ***Question 1*** *and three other questions. If you answer more than three of the questions 2 to 5, only your best three will be counted. Question 1 is worth* ***28 marks****; the other questions are each worth* ***20 marks****.*

1 (i) (a) Give the English names for the Greek letters γ and ν. ***(1 mark)***

(b) Give the symbol for the set of rational numbers. ***(1 mark)***

(ii) Let $n \in \mathbb{N}$. Consider the following six conditions:

(1) 3 divides n; (2) 9 divides n; (3) 12 divides n;
(4) $n = 24$; (5) 3 divides n^2; (6) n is even and 3 divides n.

Which of the conditions imply that $n \in \mathbb{N}$ is divisible by 6? Which of the conditions are implied by $n \in \mathbb{N}$ is divisible by 6? ***(6 marks)***

(iii) (a) What does it mean for a natural number n to be a *composite* number? ***(1 mark)***

(b) Show that 1001 is composite. ***(2 marks)***

(c) What is the *canonical prime factorisation* of a natural number n? Determine the canonical prime factorisation of 364364, using the previous part if necessary. ***(4 marks)***

(iv) State the *division algorithm* for polynomials in $\mathbb{R}[x]$. Use it to determine the remainder when $x^5 - 3x^3 - 3x^2 + 2x$ is divided by $x^2 - x - 2$. ***(6 marks)***

(v) State a simple condition in terms of f and its derivative f' for a number a to be a repeated root of a polynomial f. Use it to find a repeated root of the polynomial

$$x^4 - 2x^3 - 4x^2 + 2x + 3,$$

and hence determine all of its roots. ***(7 marks)***

2 (i) What is meant by the *highest common factor* (a, b) of integers a and b, not both zero? ***(2 marks)***

(ii) (a) Find the smallest natural number b for which one can find integers x and y satisfying the equation $616x + 231y = 10000 + b$. ***(3 marks)***

(b) With this value of b, how many solutions are there to the equation $616x + 231y = 10000 + b$ with both x and y *natural numbers*? Give the solution with x as small a natural number as possible. ***(7 marks)***

(iii) A student buys a round of drinks for £2.10 each and some bags of crisps for 55 pence each. The total cost is £22.85. How many of each did he buy? ***(8 marks)***

3 (i) Find all solutions to the congruences $78x \equiv 104 \pmod{143}$ and $78x \equiv 105 \pmod{143}$, expressing your answers in the form $x \equiv a \pmod{m}$. ***(6 marks)***

(ii) Find the general simultaneous solution to

$$x \equiv 2 \pmod{11} \qquad \text{and} \qquad x \equiv 3 \pmod{13},$$

expressing your answers in the form $x \equiv a \pmod{m}$. ***(3 marks)***

(iii) Find an integer which is three times a square and twice a cube. ***(4 marks)***

(iv) The four digit number $n = abcd$ represents $10^3a + 10^2b + 10c + d$. Using congruences modulo 11, show that n is divisible by 11 if and only if the alternating sum of digits $a - b + c - d$ is divisible by 11.

State a generalisation to numbers with n digits, and also a similar criterion for divisibility by 9.

The number $16! = 2a92278988b000$ for some digits a and b. Using the earlier parts of the question, find a and b, explaining your working clearly. ***(7 marks)***

4 (i) What is $2006^{2006} \pmod{55}$? ***(6 marks)***

(ii) Prove by induction that $7(3)^n + 3(8)^n$ is divisible by 5 for all non-negative integers n.

Give another proof using congruences. ***(8 marks)***

(iii) Prove by induction that for all positive integers n, there is an n digit number, whose only digits are 1 and 2, which is divisible by 2^n. ***(6 marks)***

5 (i) Prove that $\sqrt{12}$ is irrational. ***(3 marks)***

(ii) Give decimal expansions of both a rational number and an irrational number strictly between $0.\dot{1}23456789\dot{0}$ and $0.\dot{1}2345678\dot{9}$, explaining your answer briefly.

(3 marks)

(iii) Write the real number $0.19\dot{9}6\dot{2}$ as a fraction in lowest terms. ***(7 marks)***

(iv) Write $x = 0.\dot{a}_1a_2a_3a_4a_5\dot{a}_6$, a recurring decimal with period 6. Write the decimal $y = 0.\dot{a}_6a_1a_2a_3a_4\dot{a}_5 = 0.a_6a_1a_2a_3a_4a_5a_6a_1a_2a_3a_4a_5\ldots$ in terms of a_6 and x. Suppose that $y = 4x$, and that $a_6 = 9$. Write x as a fraction.

Write down a 6 digit integer, ending with 9, which when multiplied by 4 is equal to the original number but with the final 9 moved to the front. ***(7 marks)***

End of Question Paper